

Topic 8 · Sheet A: Questions

Inclusion–exclusion and Boole’s inequality

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 7

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the inclusion–exclusion formula $\mathbb{P}(\cup_i A_i) = P_1 - P_2 + P_3 - \dots$, where P_k is the sum of the probabilities of all intersections of k of the events, the same formula for counting $|A_1 \cup \dots \cup A_n|$, and Boole’s inequality $\mathbb{P}(\cup_i A_i) \leq \sum_i \mathbb{P}(A_i)$.

Tip. “At least one” is a union; “none” is its complement. Write down $P_1, P_2, P_3, \dots$ separately before combining them.

1

4 marks

- (a) $|A| = 20$, $|B| = 15$ and $|A \cap B| = 5$. Find $|A \cup B|$. [1]
- (b) Three sets each have 10 elements, each pair shares 4, and all three share 2. How many elements are in their union? [1]
- (c) $\mathbb{P}(A) = 0.3$, $\mathbb{P}(B) = 0.4$ and $\mathbb{P}(A \cap B) = 0.1$. Find $\mathbb{P}(A \cup B)$. [1]
- (d) Three events have probabilities 0.2, 0.3 and 0.4. Give an upper bound for the probability that at least one happens. [1]

2

4 marks

Consider the integers $1, 2, \dots, 120$.

- (a) How many are divisible by 2, by 3, and by 5? [1]
- (b) How many are divisible by 6, by 10, and by 15? [1]
- (c) How many are divisible by 30? [1]
- (d) How many are divisible by at least one of 2, 3 and 5? [1]

3

6 marks

An integer is chosen at random from $1, 2, \dots, 120$. Find the probability that it is divisible by:

- (a) none of 2, 3 and 5; [2]
- (b) exactly one of 2, 3 and 5; [2]

- (c) 2, but neither 3 nor 5. [2]

4

6 marks

Of 70 students, 30 study French, 25 German and 20 Spanish. 10 study French and German, 8 French and Spanish, 6 German and Spanish, and 3 study all three.

- (a) How many study at least one of these languages? [2]
(b) How many study none of them? [1]
(c) How many study exactly one of them? [3]

5

6 marks

A die is rolled 4 times.

- (a) Let A_i be “roll i is a 6”. Use inclusion–exclusion to find the probability of at least one 6, and check it with the complement. [3]
(b) Find the probability that both a 1 and a 2 appear among the four rolls. [3]

6

6 marks

- (a) Use $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ to show that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$. [2]
(b) A machine has 10 parts, each failing with probability 0.01. Without assuming independence, give an upper bound for the probability that at least one part fails. [2]
(c) If the parts fail independently, find the exact probability, and compare. [2]

7

6 marks

Three letters are put at random into three addressed envelopes, one in each. A *derangement* is an arrangement in which no letter is in its own envelope.

- (a) List all 6 arrangements and pick out the derangements. [2]
(b) Count the derangements with inclusion–exclusion, where A_i is “letter i is in its own envelope”. [2]
(c) Find the probability of a derangement. [2]

8

6 marks

Five letters are put at random into five addressed envelopes.

- (a) Use inclusion–exclusion to find the number of derangements. [3]
(b) Find the probability that no letter is in its own envelope. [1]
(c) Compare your answer with $e^{-1} \approx 0.368$. [2]

9

6 marks

A hand of 5 cards is dealt from a shuffled deck of 52.

- (a) Find the probability that it has no hearts. [2]
- (b) Find the probability that it has no hearts and no spades. [1]
- (c) Find the probability that it has at least one heart and at least one spade. [3]

10

7 marks

Five different balls are put into three boxes, each ball into a box chosen at random.

- (a) Let A_j be “box j is empty”. Use inclusion–exclusion to count the assignments that leave no box empty. [4]
- (b) Find the probability that no box is empty. [1]
- (c) Give a formula for the number of ways to put n different balls into 3 boxes with none empty. [2]

11

8 marks

Exam style. Consider the integers $1, 2, \dots, 1000$.

- (a) How many are divisible by 2, by 3, and by 7? [2]
- (b) How many are divisible by 6, by 14, and by 21? [2]
- (c) How many are divisible by 42? [1]
- (d) How many are divisible by none of 2, 3 and 7? What is the probability that a random one of them is? [3]

12

8 marks

Exam style.

- (a) Use the formula for two events, applied to $A \cup B$ and C , to prove inclusion–exclusion for three events. [4]
- (b) Three dice are rolled. Use inclusion–exclusion to find the probability of at least one 6. [2]
- (c) Let E_{ij} be “dice i and j show the same number”. Use inclusion–exclusion to find the probability that some two dice match, and check it with a complement. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.