

Topic 8 · Sheet A: Worked answers and mark scheme

Inclusion–exclusion and Boole’s inequality

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 7

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Warm-up

4 marks

(a) $20 + 15 - 5 = 30$. (b) $30 - 12 + 2 = 20$. (c) $0.3 + 0.4 - 0.1 = 0.6$. (d) 0.9 , by Boole’s inequality.

B1 each part correct

4

Why this question In (b), elements in all three sets are added three times, subtracted three times, and so added back once at the end.

2 Counting multiples

4 marks

(a) $60, 40, 24$. (b) $20, 12, 8$. (c) 4 . (d) $124 - 40 + 4 = 88$.

B1 each part correct

4

Common mistake In (b), dividing by $2 \cdot 3 = 6$ is right, but for (c) use the lowest common multiple 30, which here equals $2 \cdot 3 \cdot 5$.

3 None, exactly one, one but not the others

6 marks

(a) $120 - 88 = 32$, so $\frac{32}{120} = \frac{4}{15}$.

(b) Exactly one $= P_1 - 2P_2 + 3P_3 = 124 - 80 + 12 = 56$, so $\frac{56}{120} = \frac{7}{15}$.

(c) $60 - 20 - 12 + 4 = 32$, so $\frac{4}{15}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question In (b), an element in two sets is counted twice in P_1 and once in P_2 , so $P_1 - 2P_2$ removes it. Elements in all three need the $+3P_3$.

4 Languages

6 marks

(a) $30 + 25 + 20 - 10 - 8 - 6 + 3 = 54$.

(b) $70 - 54 = 16$.

(c) Exactly two: $(10 + 8 + 6) - 3 \cdot 3 = 15$. Exactly three: 3. So exactly one: $54 - 15 - 3 = 36$.

M1	(a)	1
A1	(a)	1
B1	(b)	1
M1	(c) exactly two	1
M1	(c) subtracting from (a)	1
A1	(c)	1

Common mistake In (c), using $10 + 8 + 6 = 24$ as the number studying exactly two. The 3 who study all three are inside each of those pairs.

5 Four rolls

6 marks

- (a) $\binom{4}{1}\frac{1}{6} - \binom{4}{2}\frac{1}{36} + \binom{4}{3}\frac{1}{216} - \binom{4}{4}\frac{1}{1296} = \frac{864 - 216 + 24 - 1}{1296} = \frac{671}{1296}$. Check: $1 - \left(\frac{5}{6}\right)^4 = \frac{1296 - 625}{1296}$.
- (b) $\mathbb{P}(\text{no 1 or no 2}) = 2\left(\frac{5}{6}\right)^4 - \left(\frac{4}{6}\right)^4 = \frac{1250 - 256}{1296}$, so the answer is $\frac{302}{1296} = \frac{151}{648}$.

M1	(a) the four terms	1
A1	(a)	1
B1	(a) check	1
M1	(b) complement	1
M1	(b) inclusion–exclusion	1
A1	(b)	1

Why this question For “at least one” the complement is quicker, but for “both a 1 and a 2” you need inclusion–exclusion on the complement.

6 Boole’s inequality

6 marks

- (a) $\mathbb{P}(A \cap B) \geq 0$, so subtracting it can only make the right-hand side smaller.
- (b) $10 \cdot 0.01 = 0.1$.
- (c) $1 - 0.99^{10} \approx 0.0956$, just below the bound.

E2	(a)	2
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Boole’s inequality needs no independence at all, and for small probabilities it is very close to the truth.

7 Three letters

6 marks

- (a) Writing where letters 1, 2, 3 go: 123, 132, 213, 231, 312, 321. The derangements are **231 and 312**.
- (b) $|A_i| = 2!$, $|A_i \cap A_j| = 1!$, $|A_1 \cap A_2 \cap A_3| = 1$. So $|\bigcup A_i| = 3 \cdot 2 - 3 \cdot 1 + 1 = 4$, and the derangements number $6 - 4 = 2$.
- (c) $\frac{1}{3}$.

B1	(a) list	1
B1	(a) derangements	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Listing checks the formula on a small case; the formula is what scales to larger n .

8 Five letters

6 marks

- (a) $5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0! = 120 - 120 + 60 - 20 + 5 - 1 = 44$.
- (b) $\frac{44}{120} = \frac{11}{30}$, about 0.367.
- (c) **Very close**: the probability of a derangement tends to e^{-1} as the number of letters grows.

M1	(a) the terms $\binom{5}{k}(5-k)!$	1
M1	(a) alternating signs	1
A1	(a)	1
B1	(b)	1
E2	(c)	2

Common mistake Using $\binom{5}{k}$ alone for the k -fold intersections. Once k letters are fixed, the other $5-k$ can still be arranged in $(5-k)!$ ways.

9 Two suits in a hand

6 marks

- (a) $\frac{\binom{39}{5}}{\binom{52}{5}} = \frac{2109}{9520}$, about 0.222.
- (b) $\frac{\binom{26}{5}}{\binom{52}{5}} = \frac{253}{9996}$, about 0.025.
- (c) $1 - \left[2 \cdot \frac{2109}{9520} - \frac{253}{9996} \right] \approx 0.582$.

M1	(a)	1
A1	(a)	1
B1	(b)	1
M1	(c) complement	1
M1	(c) inclusion–exclusion	1
A1	(c)	1

Why this question The complement of “at least one heart and at least one spade” is “no hearts or no spades”, a union, which is where inclusion–exclusion applies.

10 No empty box

7 marks

- (a) $|A_j| = 2^5$ (every ball avoids box j), $|A_i \cap A_j| = 1^5$, and all three boxes cannot be empty. So $|\bigcup A_j| = 3 \cdot 32 - 3 \cdot 1 + 0 = 93$, and $3^5 - 93 = 150$.
- (b) $\frac{150}{243} = \frac{50}{81}$.
- (c) $3^n - 3 \cdot 2^n + 3$.

M1	(a) $ A_j = 2^5$	1
M1	(a) $ A_i \cap A_j = 1$	1
M1	(a) combining	1
A1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Check (c) with $n = 3$: $27 - 24 + 3 = 6 = 3!$, the number of ways to put three balls into three boxes one each.

11 Exam style: avoiding three divisors

8 marks

- (a) $\lfloor 1000/2 \rfloor = 500$, $\lfloor 1000/3 \rfloor = 333$, $\lfloor 1000/7 \rfloor = 142$.
- (b) $\lfloor 1000/6 \rfloor = 166$, $\lfloor 1000/14 \rfloor = 71$, $\lfloor 1000/21 \rfloor = 47$.
- (c) $\lfloor 1000/42 \rfloor = 23$.
- (d) At least one: $975 - 284 + 23 = 714$. None: $1000 - 714 = 286$, probability 0.286.

B1	(a)	1
B1	(a)	1
B1	(b)	1
B1	(b)	1
B1	(c)	1
M1	(d) inclusion–exclusion	1
A1	(d) count	1
A1	(d) probability	1

Common mistake Rounding $\frac{1000}{3}$ up to 334. Count multiples with the floor: $3 \cdot 333 = 999$ is the last multiple of 3 up to 1000.

12 Exam style: three events

8 marks

- (a) $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C)$. Then $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, so $\mathbb{P}((A \cup B) \cap C) = \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C)$. Expanding $\mathbb{P}(A \cup B)$ too gives $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$.
- (b) $3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{36} + \frac{1}{216} = \frac{91}{216}$.
- (c) Each $\mathbb{P}(E_{ij}) = \frac{1}{6}$. Any two of the E_{ij} together force all three dice equal, as does all three: each has probability $\frac{1}{36}$. So $3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{36} + \frac{1}{36} = \frac{4}{9}$. Check: $1 - \frac{6 \cdot 5 \cdot 4}{216} = \frac{96}{216} = \frac{4}{9}$.

M1	(a) two-event formula for $A \cup B$ and C	1
M1	(a) distributive law	1
M1	(a) two-event formula again	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) intersections	1
A1	(c) with check	1

Why this question In (c), the pairwise intersections all equal the triple intersection. Inclusion-exclusion still works, but noticing this structure is what makes it quick.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.