

Topic 7 · Sheet A: Questions

Event spaces and the axioms of probability

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 6

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need event spaces (which contain Ω , and are closed under complements and countable unions), the three axioms of probability, the properties that follow from them, and probabilities defined by length or area.

The axioms. $\mathbb{P}(A) \geq 0$ for every event A ; $\mathbb{P}(\Omega) = 1$; and if $A_1, A_2, \dots$ are pairwise disjoint, then $\mathbb{P}(\bigcup_n A_n) = \sum_n \mathbb{P}(A_n)$. “Uniform” on a region means probability is proportional to length or area.

1

4 marks

Let $\Omega = \{1, 2, 3, 4\}$. Is each collection an event space on Ω ?

- (a) $\{\emptyset, \Omega\}$ [1]
- (b) $\{\emptyset, \{1\}, \{2, 3, 4\}, \Omega\}$ [1]
- (c) $\{\emptyset, \{1\}, \{2\}, \Omega\}$ [1]
- (d) The set of all subsets of Ω [1]

2

4 marks

Events A and B have $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.4$ and $\mathbb{P}(A \cup B) = 0.7$. Find:

- (a) $\mathbb{P}(A \cap B)$; [1]
- (b) $\mathbb{P}(A^c)$; [1]
- (c) $\mathbb{P}(A \setminus B)$; [1]
- (d) $\mathbb{P}(A^c \cap B^c)$. [1]

3

6 marks

Let $\Omega = \{1, 2, 3, 4, 5\}$.

- (a) Find the smallest event space that contains $\{1, 2\}$. [2]
- (b) Find the smallest event space that contains both $\{1\}$ and $\{2\}$. How many sets does it have? [3]

- (c) Explain why every event space must contain $\emptyset$. [1]

4

6 marks

Using only the three axioms, show that:

- (a) $\mathbb{P}(\emptyset) = 0$; [2]
(b) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$; [2]
(c) if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$. [2]

5

6 marks

A loaded die shows k with probability ck , for $k = 1, \dots, 6$.

- (a) Find c . [2]
(b) Find the probability of an even number. [2]
(c) Find the probability of at least 5. [2]

6

6 marks

Let $\Omega = \{1, 2, 3, \dots\}$, with every subset an event, and $\mathbb{P}(\{n\}) = c \left(\frac{1}{3}\right)^n$.

- (a) Find c . [2]
(b) Find the probability that the outcome is at most 3. [2]
(c) Find the probability that the outcome is odd. [2]

7

6 marks

Decide whether each rule defines a probability measure. If not, name an axiom that fails.

- (a) $\Omega = \{a, b, c\}$, with $\mathbb{P}(\{a\}) = 0.5$, $\mathbb{P}(\{b\}) = 0.3$, $\mathbb{P}(\{c\}) = 0.3$, extended by adding. [2]
(b) $\Omega = \{a, b, c\}$, with $\mathbb{P}(A) = \frac{|A|^2}{9}$ for every subset A . [2]
(c) $\Omega = \{1, 2, 3, 4\}$, events $\{\emptyset, \{1, 2\}, \{3, 4\}, \Omega\}$, with $\mathbb{P}(\{1, 2\}) = 0.3$, $\mathbb{P}(\{3, 4\}) = 0.7$. [2]

8

6 marks

A point (X, Y) is chosen uniformly in the unit square $[0, 1]^2$. Find:

- (a) $\mathbb{P}(X + Y \leq 1)$; [2]
(b) $\mathbb{P}(X > 2Y)$; [2]
(c) the probability that both coordinates are at most $\frac{1}{2}$; [1]
(d) $\mathbb{P}(X = Y)$. [1]

9

6 marks

A point is chosen uniformly in the disc of radius 1 centred at the origin. Find the probability that:

- (a) its distance from the origin is less than $\frac{1}{2}$; [2]
- (b) it lies in the first quadrant; [1]
- (c) its distance from the origin is at least $\frac{1}{2}$ and its x -coordinate is positive; [2]
- (d) it lies exactly on the circle of radius $\frac{1}{2}$. [1]

10

7 marks

Let $\Omega = \{\omega_1, \dots, \omega_n\}$, with every subset an event and all the sets $\{\omega_k\}$ equally likely.

- (a) Use the axioms to show that $\mathbb{P}(\{\omega_k\}) = \frac{1}{n}$. [3]
- (b) Hence show that $\mathbb{P}(A) = \frac{|A|}{n}$ for every event A . [2]
- (c) Explain why this argument breaks down if Ω is countably infinite. [2]

11

8 marks

Exam style. A loaded four-sided die has $\Omega = \{1, 2, 3, 4\}$. We know that $\mathbb{P}(\{1, 2\}) = 0.3$, $\mathbb{P}(\{2, 3\}) = 0.5$ and $\mathbb{P}(\{1, 2, 3\}) = 0.6$.

- (a) Find $\mathbb{P}(\{k\})$ for each k . [4]
- (b) Find the probability of an even number. [1]
- (c) Find $\mathbb{P}(\{1, 4\})$. [1]
- (d) Find the probability of an odd number, given that the result is at least 2. [2]

12

8 marks

Exam style. Two numbers X and Y are chosen independently and uniformly from $[0, 1]$, so (X, Y) is uniform on the unit square.

- (a) Find $\mathbb{P}(Y > X^2)$. [2]
- (b) Find $\mathbb{P}(XY < \frac{1}{2})$. [3]
- (c) Find $\mathbb{P}(X^2 + Y^2 \leq 1)$. [2]
- (d) $X = Y$ is possible, yet $\mathbb{P}(X = Y) = 0$. Explain why this does not contradict the axioms. [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.