

Topic 7 · Sheet A: Worked answers and mark scheme

Event spaces and the axioms of probability

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 6

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Event spaces

4 marks

(a) **Yes.** (b) **Yes.** (c) **No:** $\{1\}^c = \{2, 3, 4\}$ is missing (so is $\{1\} \cup \{2\}$). (d) **Yes.**

B1 each part correct

4

Why this question For a finite Ω , checking complements and pairwise unions is enough. (a) is the smallest event space and (d) the largest.

2 Consequences of the axioms

4 marks

(a) $0.5 + 0.4 - 0.7 = 0.2$. (b) 0.5 . (c) $0.5 - 0.2 = 0.3$. (d) $1 - 0.7 = 0.3$.

B1 each part correct

4

Common mistake In (c), using $\mathbb{P}(A) - \mathbb{P}(B)$. That only works when $B \subseteq A$; in general subtract $\mathbb{P}(A \cap B)$.

3 Generating event spaces

6 marks

- (a) $\{\emptyset, \{1, 2\}, \{3, 4, 5\}, \Omega\}$.
- (b) It must contain $\{1\}$, $\{2\}$ and their complements, and all unions. The pieces $\{1\}$, $\{2\}$, $\{3, 4, 5\}$ split Ω , and the event space is every union of these pieces: $\emptyset, \{1\}, \{2\}, \{3, 4, 5\}, \{1, 2\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \Omega$. **8 sets**.
- (c) Ω is in it, so its complement $\emptyset$ is too.

B1	(a)	1
E1	(a) closed under complements and unions	1
M1	(b) identifying the three pieces	1
A1	(b) all unions	1
B1	(b) 8	1
E1	(c)	1

Why this question An event space on a finite set is built from a partition: its sets are all the unions of the parts, 2^k of them for k parts.

4 Proofs from the axioms

6 marks

- (a) Take $A_1 = \Omega$ and $A_2 = A_3 = \dots = \emptyset$, which are pairwise disjoint with union Ω . Countable additivity gives $1 = 1 + \sum_{n \geq 2} \mathbb{P}(\emptyset)$, so $\mathbb{P}(\emptyset) = 0$.
- (b) A and A^c are disjoint with union Ω , so $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$ (using additivity with empty sets after them).
- (c) B is the disjoint union of A and $B \setminus A$, so $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$, since probabilities are non-negative.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) splitting B	1
A1	(c) non-negativity	1

Why this question Every familiar rule of probability follows from three short axioms. That is what makes them the right foundation.

5 A loaded die

6 marks

(a) $c(1 + 2 + \cdots + 6) = 21c = 1$, so $c = \frac{1}{21}$.

(b) $\frac{2+4+6}{21} = \frac{4}{7}$.

(c) $\frac{5+6}{21} = \frac{11}{21}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question On a finite set, choosing non-negative probabilities for single outcomes that add to 1 always gives a probability measure.

6 A countable sample space

6 marks

(a) $\sum_{n \geq 1} \left(\frac{1}{3}\right)^n = \frac{1/3}{1-1/3} = \frac{1}{2}$, so $c = 2$.

(b) $2 \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right) = \frac{26}{27}$.

(c) $2 \sum_{k \geq 0} \left(\frac{1}{3}\right)^{2k+1} = \frac{2/3}{1-1/9} = \frac{3}{4}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake In (c), answering $\frac{1}{2}$ because “half the numbers are odd”. The outcomes are not equally likely; small numbers carry most of the probability.

7 Is it a probability measure?

6 marks

(a) **No:** $\mathbb{P}(\Omega) = 1.1 \neq 1$.

(b) **No:** $\mathbb{P}(\Omega) = 1$, but $\mathbb{P}(\{a\}) + \mathbb{P}(\{b\}) = \frac{2}{9} \neq \frac{4}{9} = \mathbb{P}(\{a, b\})$, so **additivity fails**.

(c) **Yes:** the four events are closed under complements and unions, every value is non-negative, $\mathbb{P}(\Omega) = 1$, and $\mathbb{P}(\{1, 2\}) + \mathbb{P}(\{3, 4\}) = \mathbb{P}(\Omega)$.

B1	(a)	1
E1	(a)	1
B1	(b)	1
E1	(b)	1
B1	(c)	1
E1	(c)	1

Why this question In (c), $\{1\}$ is not an event at all, so it has no probability. A measure only needs values on the event space.

8 Areas in a square

6 marks

- (a) The triangle below $x + y = 1$: $\frac{1}{2}$.
- (b) The triangle with corners $(0,0)$, $(1,0)$, $(1, \frac{1}{2})$: $\frac{1}{4}$.
- (c) The square $[0, \frac{1}{2}]^2$: $\frac{1}{4}$.
- (d) A line has area 0: 0.

M1	(a)	1
A1	(a)	1
M1	(b) the region	1
A1	(b)	1
B1	(c)	1
B1	(d)	1

Common mistake In (b), using the line $y = 2x$. Rearranging $X > 2Y$ gives $Y < \frac{X}{2}$, a much thinner triangle.

9 A point in a disc

6 marks

- (a) $\frac{\pi(1/2)^2}{\pi} = \frac{1}{4}$.
- (b) $\frac{1}{4}$.
- (c) The ring has probability $1 - \frac{1}{4} = \frac{3}{4}$, and half of it has $x > 0$: $\frac{3}{8}$.
- (d) A circle has area 0: 0.

M1	(a) ratio of areas	1
A1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Why this question Half the radius gives only a quarter of the area, so most of a uniform disc lies near its edge.

10 Deriving equally likely outcomes

7 marks

- (a) The sets $\{\omega_1\}, \dots, \{\omega_n\}$ are disjoint with union Ω . By finite additivity, $\sum_k \mathbb{P}(\{\omega_k\}) = \mathbb{P}(\Omega) = 1$. They are all equal, say to p , so $np = 1$ and $p = \frac{1}{n}$.
- (b) A is the disjoint union of the $|A|$ singletons it contains, so $\mathbb{P}(A) = |A| \cdot \frac{1}{n}$.
- (c) Equal probabilities p on infinitely many outcomes would add to 0 (if $p = 0$) or to infinity (if $p > 0$), **never** 1. So equally likely outcomes are impossible.

M1	(a) singletons partition Ω	1
M1	(a) additivity	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
E2	(c)	2

Why this question The school rule “favourable over possible” is a theorem, not a definition. It holds exactly when the space is finite and the outcomes are equally likely.

11 Exam style: setting up a measure

8 marks

- (a) $\mathbb{P}(\{4\}) = 1 - \mathbb{P}(\{1, 2, 3\}) = 0.4$. $\mathbb{P}(\{3\}) = 0.6 - 0.3 = 0.3$. $\mathbb{P}(\{2\}) = 0.5 - 0.3 = 0.2$.
 $\mathbb{P}(\{1\}) = 0.3 - 0.2 = 0.1$.
- (b) $0.2 + 0.4 = 0.6$.
- (c) $0.1 + 0.4 = 0.5$.
- (d) $\frac{\mathbb{P}(\{3\})}{\mathbb{P}(\{2, 3, 4\})} = \frac{0.3}{0.9} = \frac{1}{3}$.

B1	(a) $\mathbb{P}(\{4\})$	1
B1	(a) $\mathbb{P}(\{3\})$	1
B1	(a) $\mathbb{P}(\{2\})$	1
B1	(a) $\mathbb{P}(\{1\})$	1
B1	(b)	1
B1	(c)	1
M1	(d)	1
A1	(d)	1

Common mistake Not checking the answer. The four values are non-negative and add up to 1, so they really do define a probability measure.

12 Exam style: regions in the square

8 marks

- (a) The area under $y = x^2$ is $\int_0^1 x^2 dx = \frac{1}{3}$, so $\frac{2}{3}$.
- (b) For $x \leq \frac{1}{2}$ every y works; for $x > \frac{1}{2}$ we need $y < \frac{1}{2x}$. Area $\frac{1}{2} + \int_{1/2}^1 \frac{1}{2x} dx = \frac{1}{2} + \frac{1}{2} \ln 2$, about 0.847.
- (c) A quarter of the unit disc: $\frac{\pi}{4}$.
- (d) The axioms only say probabilities are non-negative, add up correctly and total 1. **Probability 0 does not mean impossible**: a line has no area, even though points on it can occur.

M1	(a)	1
A1	(a)	1
M1	(b) splitting at $x = \frac{1}{2}$	1
M1	(b) integral	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
E1	(d)	1

Why this question With uniform points, probability questions become area questions, and integrals compute areas of curved regions.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.