

Topic 6 · Sheet A: Worked answers and mark scheme

Counting: permutations, combinations and arrangements

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 5

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Order or no order

4 marks

(a) $5! = 120$. (b) $7 \cdot 6 \cdot 5 = 210$. (c) $\binom{7}{3} = 35$. (d) $10^4 = 10\,000$.

B1 each part correct

4

Why this question (b) and (c) choose the same three runners; medals make the order matter, so (b) is $3! = 6$ times (c).

2 The multiplication rule

4 marks

(a) $3 \cdot 4 \cdot 2 = 24$. (b) $26^2 \cdot 10^3 = 676\,000$. (c) $10 \cdot 9 \cdot 8 \cdot 7 = 5040$. (d) $2^8 = 256$.

B1 each part correct

4

Why this question Each is a sequence of choices, and the number of options at each step does not depend on the earlier choices, only on how many there were.

3 Words with repeated letters

6 marks

- (a) 6 letters with A three times and N twice: $\frac{6!}{3!2!} = \mathbf{60}$.
- (b) 10 letters: S three times, T three times, I twice: $\frac{10!}{3!3!2!} = \mathbf{50\,400}$.
- (c) With B first, arrange ANANA: $\frac{5!}{3!2!} = 10$. So $\frac{10}{60} = \frac{1}{6}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question (c) also follows by symmetry: each of the 6 positions is equally likely to hold the single B.

4 Committees

6 marks

- (a) $\binom{11}{4} = \mathbf{330}$.
- (b) $\binom{5}{2}\binom{6}{2} = 10 \cdot 15 = \mathbf{150}$.
- (c) All-male committees: $\binom{6}{4} = 15$. So $330 - 15 = \mathbf{315}$.
- (d) $\frac{150}{330} = \frac{5}{11}$.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) complement	1
A1	(c)	1
B1	(d)	1

Common mistake In (c), choosing one woman first (5 ways) and then any 3 others ($\binom{10}{3}$). Committees with two or more women get counted several times.

5 Card hands

6 marks

(a) $\binom{52}{5} = 2\,598\,960$.

(b) $\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{1287}{2\,598\,960} = \frac{33}{66\,640}$.

(c) Choose 2 aces and 3 non-aces: $\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{6 \cdot 17\,296}{2\,598\,960} = \frac{2162}{54\,145}$, about 0.040.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) $\binom{4}{2}$	1
M1	(c) $\binom{48}{3}$ for the others	1
A1	(c)	1

Common mistake In (c), using $\binom{50}{3}$ for the other three cards. They must not be aces, or the hand could have more than two.

6 Stars and bars

6 marks

(a) 6 stars and 2 bars: $\binom{8}{2} = 28$.

(b) Give each child one sweet first, then share the other 3: $\binom{5}{2} = 10$.

(c) 10 stars and 3 bars: $\binom{13}{3} = 286$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Identical objects into labelled boxes is an unordered sample with replacement: each child is a box, and each sweet chooses a box.

7 Four kinds of sample

6 marks

(a) $5^3 = 125$. (b) $5 \cdot 4 \cdot 3 = 60$. (c) $\binom{5}{3} = 10$. (d) $\binom{5+3-1}{3} = \binom{7}{3} = 35$.

B1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Common mistake In (d), dividing (a) by $3!$. Samples with repeats, such as $(1, 1, 2)$, have fewer than $3!$ orders, so that division is wrong.

8 The lottery

6 marks

(a) $\binom{49}{6} = 13\,983\,816$.

(b) $\frac{1}{13\,983\,816}$.

(c) $\frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} = \frac{20 \cdot 12\,341}{13\,983\,816} \approx 0.0177$.

(d) $\frac{\binom{43}{6}}{\binom{49}{6}} \approx 0.436$.

B1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question Split the 49 numbers into your 6 and the other 43; a draw with exactly k matches picks k from one group and $6 - k$ from the other.

9 Identities by counting

6 marks

(a) Choosing the 3 objects to take is the same as choosing the 7 to leave. **Both equal 120.**

(b) $56 = 21 + 35$. Fix one object: the 3-sets that contain it number $\binom{7}{2}$, and those that do not number $\binom{7}{3}$.

- (c) Both sides count all subsets of a 5-element set: by size on the left, and by **2 choices (in or out) for each element**, $2^5 = 32$, on the right.

E2	(a)	2
B1	(b) check	1
E1	(b) explanation	1
E2	(c)	2

Why this question Counting the same set in two ways proves an identity without any algebra.

10 Passwords

7 marks

- (a) $36^6 = 2\,176\,782\,336$.
- (b) Subtract the all-letter passwords: $36^6 - 26^6 = 1\,867\,866\,560$.
- (c) Choose the two digit positions, then fill them: $\binom{6}{2} \cdot 10^2 \cdot 26^4 = 685\,464\,000$.
- (d) $\frac{685\,464\,000}{2\,176\,782\,336} \approx 0.315$.

B1	(a)	1
M1	(b) complement	1
A1	(b)	1
M1	(c) positions, then characters	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Common mistake In (c), forgetting $\binom{6}{2}$. The two digits can be in any two of the six positions.

11 Exam style: poker hands

8 marks

- (a) Choose the rank for three (13) and its three suits ($\binom{4}{3}$); then the rank for two (12) and its suits ($\binom{4}{2}$): $13 \cdot 4 \cdot 12 \cdot 6 = 3744$. Probability $\frac{3744}{2\,598\,960} = \frac{6}{4165}$.
- (b) Choose the two pair ranks together ($\binom{13}{2}$, since their order does not matter), their suits ($\binom{4}{2}^2$), and the fifth card from the 44 of other ranks: $78 \cdot 36 \cdot 44 = 123\,552$. Probability $\frac{198}{4165}$, about 0.048.

M1	(a) ranks in order $13 \cdot 12$	1
M1	(a) suits $\binom{4}{3}\binom{4}{2}$	1
A1	(a) count	1
A1	(a) probability	1
M1	(b) $\binom{13}{2}$ for the pair ranks	1
M1	(b) suits and fifth card	1
A1	(b) count	1
A1	(b) probability	1

Common mistake In (b), using $13 \cdot 12$ for the pair ranks. The two pairs play the same role, so that counts every hand twice. In (a) the triple and the pair play different roles, so $13 \cdot 12$ is right there.

12 Exam style: six dice

8 marks

- (a) $\frac{6!}{6^6} = \frac{720}{46\,656} = \frac{5}{324}$.
- (b) $\frac{\binom{6}{2} 5^4}{6^6} = \frac{9375}{46\,656} = \frac{3125}{15\,552}$.
- (c) Choose the three values ($\binom{6}{3} = 20$), then arrange two of each: $\frac{6!}{2!2!2!} = 90$. So $\frac{1800}{46\,656} = \frac{25}{648}$.
- (d) Unordered with replacement, 6 from 6 values: $\binom{11}{6} = 462$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) choosing the values	1
M1	(c) arranging with repeats	1
A1	(c)	1
B1	(d)	1

Why this question The 462 unordered outcomes in (d) are not equally likely, so probabilities must still be found with the 6^6 ordered ones.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.