

Topic 4 · Sheet A: Worked answers and mark scheme

Independence

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 3

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Checking the definition

4 marks

(a) $\mathbb{P}(A \cap B) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$: **yes**. (b) $\frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2}$: **no**. (c) $\frac{1}{3} \neq \frac{1}{3} \cdot \frac{1}{2}$: **no**. (d) **No**: $\mathbb{P}(A \cap B) = 0$ but $\mathbb{P}(A)\mathbb{P}(B) > 0$.

B1 each part correct

4

Common mistake Confusing “independent” with “cannot happen together”. Disjoint events with positive probabilities are as dependent as can be: one happening rules the other out.

2 Using independence

4 marks

(a) **0.18**. (b) $0.3 + 0.6 - 0.18 = \mathbf{0.72}$. (c) $0.3 \cdot 0.4 = \mathbf{0.12}$. (d) $0.7 \cdot 0.4 = \mathbf{0.28}$.

B1 each part correct

4

Why this question (d) is $1 -$ (b), a good check that the numbers fit together.

3 Complements of independent events

6 marks

- (a) A is the disjoint union of $A \cap B$ and $A \cap B^c$, so $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A) \mathbb{P}(B) = \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A) \mathbb{P}(B^c)$.
- (b) Apply (a) to the independent pair B^c and A (in either order): then B^c and A^c are independent.
- (c) $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(A^c) \mathbb{P}(B^c) = 1 - (1 - \mathbb{P}(A))(1 - \mathbb{P}(B))$.

M1	(a) splitting A	1
M1	(a) using independence	1
A1	(a)	1
M1	(b) applying (a) again	1
A1	(b)	1
B1	(c)	1

Why this question If knowing B tells you nothing about A , then knowing “not B ” cannot tell you anything either.

4 Series and parallel

6 marks

- (a) $0.9 \cdot 0.8 = 0.72$.
- (b) $1 - 0.1 \cdot 0.2 = 0.98$.
- (c) $0.98 \cdot 0.95 = 0.931$.

M1	(a)	1
A1	(a)	1
M1	(b) complement	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake In (b), adding $0.9 + 0.8$. Both parts can work at once; use the complement “both fail” instead.

5 Repeated trials

6 marks

- (a) $1 - \left(\frac{5}{6}\right)^4 = \frac{671}{1296} \approx 0.518$.
- (b) $1 - \left(\frac{35}{36}\right)^{24} \approx 0.491$.
- (c) (a).

(d) Need $(\frac{5}{6})^n < 0.1$, so $n > \frac{\ln 0.1}{\ln(5/6)} \approx 12.6$: $n = 13$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
B1	(d)	1

Why this question These two bets puzzled gamblers in the 1600s and helped start probability theory: both seem to be “a $\frac{1}{6}$ chance, enough times”, but they differ.

6 Independence from counts

6 marks

- (a) $\mathbb{P}(S \cap P) = 0.12 = 0.4 \cdot 0.3$: **independent**.
- (b) $\mathbb{P}(S \mid P) = \frac{15}{30} = 0.5 \neq 0.4$: **not independent**. Physics students are more likely to take Statistics.
- (c) $100 \cdot 0.5 \cdot 0.3 = 15$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Independence means the proportion taking Statistics is the same among Physics students as among everyone.

7 How many equations?

6 marks

- (a) **4**: three pairs and the triple.
- (b) One for every sub-collection of at least 2 events: $2^n - n - 1$.
- (c) Each $\mathbb{P}(A_i) = \frac{1}{2}$; each pair has probability $\frac{1}{4}$ (two specified tosses); the triple has $\frac{1}{8}$. All four equations hold.
- (d) $2^4 - 4 - 1 = \mathbf{11}$.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) pairs	1
A1	(c) triple	1
B1	(d)	1

Common mistake Checking only $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$. That one equation is not enough; every pair must be checked too.

8 Other ways to spot independence

6 marks

- (a) $\mathbb{P}(A \mid B) = \mathbb{P}(A)$, so **independent**. Then $\mathbb{P}(A \cup B) = 0.4 + 0.5 - 0.2 = \mathbf{0.7}$.
- (b) $\mathbb{P}(A \cap B) = 0.5 + 0.6 - 0.8 = 0.3 = 0.5 \cdot 0.6$: **independent**.
- (c) Need $\mathbb{P}(A) = \mathbb{P}(A)^2$, so $\mathbb{P}(A) = \mathbf{0}$ or $\mathbb{P}(A) = \mathbf{1}$.

B1	(a)	1
B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Only events that are certain or impossible are independent of themselves: anything else is informative about itself.

9 Independent tosses and events

6 marks

- (a) $2 \left(\frac{1}{2}\right)^5 = \frac{1}{16}$.
- (b) At least two heads: $1 - \frac{1+5}{32} = \frac{26}{32}$. Remove all heads: $\frac{26-1}{32} = \frac{25}{32}$.

(c) By independence of the complements, $(1 - \frac{1}{n})^n$, which tends to $e^{-1} \approx 0.368$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
B1	(c) limit	1

Why this question Part (c) is surprising: n independent chances of $\frac{1}{n}$ still leave about a 37% chance that none happens.

10 Pairwise but not mutual

7 marks

- (a) All three are $\frac{1}{6}$.
- (b) $A \cap B$, $A \cap C$ and $B \cap C$ are each just the outcome $(4, 3)$, with probability $\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$. **All three pairs are independent.**
- (c) Also just $(4, 3)$: $\frac{1}{36}$, not $\frac{1}{216}$.
- (d) **No.** Once you know any two of them happened, the third is certain.

B1	(a)	1
M1	(b)	1
A1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
E1	(d)	1

Why this question Each event on its own says nothing about another, but two together say everything about the third. That is the gap between pairwise and mutual independence.

11 Exam style: archers

8 marks

- (a) $\binom{3}{2}(0.7)^2(0.3) + (0.7)^3 = 0.441 + 0.343 = 0.784$.
- (b) $(0.3)^2(0.7) = 0.063$.
- (c) A round of two misses has probability $0.3 \cdot 0.5 = 0.15$. A wins in round k with probability $0.15^{k-1} \cdot 0.7$, so $\mathbb{P}(A) = \frac{0.7}{1 - 0.15} = \frac{14}{17}$.
- (d) **No:** if A hits with the first shot, B cannot hit first.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) rounds of two misses	1
M1	(c) geometric series	1
A1	(c)	1
B1	(d)	1

Common mistake In (a), forgetting the factor $\binom{3}{2}$. Exactly two hits can happen on shots 1, 2 or 1, 3 or 2, 3.

12 Exam style: a small network

8 marks

- (a) p^2 .
- (b) $1 - (1 - p^2)(1 - p)$.
- (c) $1 - 0.19 \cdot 0.1 = 0.981$.
- (d) “c works” is contained in “connected”, so the answer is $\frac{0.9}{0.981} = \frac{100}{109}$.
- (e) Routes 1 and 2 use different links, so **they are independent**. Route 1 working forces a connection, so $\mathbb{P}(\text{connected} \mid \text{route 1}) = 1 \neq 0.981$: **not independent**.

B1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
M1	(d)	1
A1	(d)	1
B1	(e) first pair	1
B1	(e) second pair	1

Why this question Events built from different independent parts are independent. Events where one contains the other almost never are.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.