

Topic 3 · Sheet A: Worked answers and mark scheme

The law of total probability and Bayes' theorem

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 2

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 The two-event case

4 marks

(a) $0.5 \cdot 0.3 + 0.2 \cdot 0.7 = 0.29$. (b) $0.5 \cdot 0.3 = 0.15$. (c) $\frac{0.15}{0.29} = \frac{15}{29}$. (d) $\frac{14}{29}$.

B1 each part correct

4

Why this question B and B^c always form a partition, so this is the simplest case of both results.

2 Partitions

4 marks

(a) **Yes**. (b) **No**: 3 is in two of the events. (c) **Yes**. (d) **No**: 5 and 6 are in none of them.

B1 each part correct

4

Common mistake Checking only for overlaps. A partition must also cover every outcome, and each event must be possible.

3 Three machines

6 marks

(a) $0.5 \cdot 0.01 + 0.3 \cdot 0.02 + 0.2 \cdot 0.05 = 0.005 + 0.006 + 0.010 = 0.021$.

(b) $\frac{0.010}{0.021} = \frac{10}{21}$.

(c) $\frac{0.5 \cdot 0.99}{1 - 0.021} = \frac{0.495}{0.979} = \frac{45}{89}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) using $\mathbb{P}(\text{not faulty} \mid M_1) = 0.99$	1
A1	(c)	1

Why this question The third machine makes only a fifth of the items but almost half of the faulty ones. Bayes' theorem measures exactly this kind of shift.

4 Choosing a bag

6 marks

(a) $\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{1}{5} = \frac{2}{5}$.

(b) $\frac{3/10}{2/5} = \frac{3}{4}$.

(c) $\mathbb{P}(\text{blue}) = \frac{3}{5}$, and bag 1 and blue has probability $\frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$. So $\frac{1}{3}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Before the draw each bag has probability $\frac{1}{2}$. A red ball makes bag 1 more likely, and a blue ball makes it less likely.

5 A screening test

6 marks

(a) $0.02 \cdot 0.9 + 0.98 \cdot 0.05 = 0.018 + 0.049 = 0.067$.

(b) $\frac{0.018}{0.067} = \frac{18}{67}$, about 0.27.

$$(c) \frac{0.02 \cdot 0.1}{1 - 0.067} = \frac{0.002}{0.933} = \frac{2}{933}.$$

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake Answering 0.9 in (b). That is $\mathbb{P}(\text{positive} \mid \text{condition})$, the other way round. Most positives come from the much larger healthy group.

6 Moving a ball between bags

6 marks

- (a) If a white ball moves, B has 5 white of 6; if black moves, 4 white of 6. $\frac{2}{5} \cdot \frac{5}{6} + \frac{3}{5} \cdot \frac{4}{6} = \frac{10+12}{30} = \frac{11}{15}$.
- (b) $\frac{10/30}{22/30} = \frac{5}{11}$.

M1	(a) partition by the ball moved	1
M1	(a) contents of B in each case	1
A1	(a)	1
M1	(b) Bayes	1
M1	(b) numerator $\frac{2}{5} \cdot \frac{5}{6}$	1
A1	(b)	1

Why this question Moving a white ball makes a white second draw more likely, so seeing white raises the chance from $\frac{2}{5}$ to $\frac{5}{11}$.

7 Weather and buses

6 marks

(a) $0.2 \cdot 0.4 + 0.5 \cdot 0.1 + 0.3 \cdot 0.05 = 0.08 + 0.05 + 0.015 = 0.145.$

(b) $\frac{0.08}{0.145} = \frac{16}{29}.$

(c) $\frac{0.3 \cdot 0.95}{0.855} = \frac{0.285}{0.855} = \frac{1}{3}.$

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Rain only happens one day in five, but more than half of late buses happen on rainy days.

8 A double-headed coin

6 marks

(a) $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}.$

(b) $\frac{1/2}{3/4} = \frac{2}{3}.$

(c) $\frac{\frac{1}{2} \cdot 1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{4}} = \frac{4}{5}.$

(d) $\frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right)^n} = \frac{2^n}{2^n + 1}.$

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Why this question Each head makes the double-headed coin more likely, but it never becomes certain: a fair coin can always produce a run of heads.

9 A die, then coins

6 marks

(a) $\sum_{k=1}^6 \frac{1}{6} \left(\frac{1}{2}\right)^k = \frac{1}{6} \left(1 - \frac{1}{64}\right) = \frac{21}{128}$.

(b) $\frac{\frac{1}{6} \cdot \frac{1}{2}}{21/128} = \frac{32}{63}$.

M1	(a) partition by the die	1
M1	(a) geometric sum	1
A1	(a)	1
M1	(b) Bayes	1
M1	(b) numerator $\frac{1}{12}$	1
A1	(b)	1

Common mistake In (b), answering $\frac{1}{6}$. Seeing no heads is much more likely after few tosses, so it makes a small die result far more likely.

10 An unseen first ball

7 marks

(a) $\frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{6}{20} + \frac{6}{20} = \frac{3}{5}$.

(b) $\frac{6/20}{12/20} = \frac{1}{2}$.

(c) Since nobody looks at the first ball, **nothing distinguishes the two positions**: each ball is equally likely to be second as first.

M1	(a) partition by the first ball	1
M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
E2	(c)	2

Why this question Throwing away a ball unseen gives no information, so it cannot change the chance of red.

11 Exam style: which coin?

8 marks

- (a) $\frac{1}{3}(0.5^2 + 0.8^2 + 0.2^2) = \frac{1}{3}(0.25 + 0.64 + 0.04) = 0.31$.
- (b) $\frac{0.64/3}{0.31} = \frac{0.64}{0.93} = \frac{64}{93}$.
- (c) Weight each coin's $\mathbb{P}(H)$ by its probability given HH : $\frac{0.25 \cdot 0.5 + 0.64 \cdot 0.8 + 0.04 \cdot 0.2}{0.93} = \frac{0.645}{0.93} = \frac{43}{62}$, about 0.69.
- (d) The first toss is a head with probability $\frac{1}{3}(0.5 + 0.8 + 0.2) = \frac{1}{2}$. **Two heads raise the chance of another to about 0.69.**

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) law of total probability with updated weights	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Common mistake In (c), using the original weights $\frac{1}{3}$. After seeing HH , the coins are no longer equally likely; use the probabilities given HH .

12 Exam style: testing twice

8 marks

- (a) $\frac{0.01 \cdot 0.9}{0.01 \cdot 0.9 + 0.99 \cdot 0.1} = \frac{0.009}{0.108} = \frac{1}{12}$.
- (b) $\frac{0.01 \cdot 0.81}{0.01 \cdot 0.81 + 0.99 \cdot 0.01} = \frac{0.0081}{0.0180} = \frac{9}{20}$.
- (c) A false positive happens 10% of the time, but two false positives only 1% of the time, while a true positive repeats 81% of the time. So **two positives are much stronger evidence**, and the probability rises from about 0.08 to 0.45.

M1	(a) Bayes	1
M1	(a) denominator	1
A1	(a)	1
M1	(b) squaring the conditional probabilities	1
M1	(b)	1
A1	(b)	1
E2	(c)	2

Why this question You can also do (b) in two steps: use $\frac{1}{12}$ from (a) as the new starting probability, then apply Bayes' theorem once more. You get the same answer.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.