

Topic 1 · Sheet A: Worked answers and mark scheme

Probability and problem solving

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Nothing earlier

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Two dice

4 marks

(a) $\frac{6}{36} = \frac{1}{6}$. (b) $\frac{6}{36} = \frac{1}{6}$. (c) $1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$. (d) $(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)$: $\frac{1}{6}$.

B1 each part correct

4

Why this question With 36 equally likely ordered pairs, every question about two dice is a count. Listing the pairs is safest when the count is small.

2 Four tosses

4 marks

(a) $\frac{\binom{4}{2}}{16} = \frac{3}{8}$. (b) $1 - \frac{1}{16} = \frac{15}{16}$. (c) T, T, H : $\frac{1}{8}$. (d) $\frac{1}{16}$.

B1 each part correct

4

Common mistake In (c), answering $\frac{1}{16}$. Only the first three tosses matter; the fourth can be anything.

3 Drawing without replacement

6 marks

- (a) $\frac{5}{8} \cdot \frac{4}{7} = \frac{5}{14}$.
- (b) Red then blue, or blue then red: $\frac{5}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{28}$.
- (c) $\frac{5}{8} \cdot \frac{4}{7} + \frac{3}{8} \cdot \frac{5}{7} = \frac{20+15}{56} = \frac{5}{8}$.

M1	(a)	1
A1	(a)	1
M1	(b) both orders	1
A1	(b)	1
M1	(c) split on the first ball	1
A1	(c)	1

Why this question The second ball is red with the same probability as the first. Before you look at the first ball, nothing distinguishes the two positions.

4 Waiting for a six

6 marks

- (a) $n - 1$ non-sixes, then a six: $\left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$.
- (b) $1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$.
- (c) $\sum_{k \geq 1} \left(\frac{5}{6}\right)^{2k-1} \frac{1}{6} = \frac{5}{36} \sum_{k \geq 0} \left(\frac{25}{36}\right)^k = \frac{5}{36} \cdot \frac{36}{11} = \frac{5}{11}$.

M1	(a)	1
A1	(a)	1
M1	(b) complement	1
A1	(b)	1
M1	(c) geometric series with ratio $\frac{25}{36}$	1
A1	(c)	1

Common mistake In (b), adding $\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$. The events “six on roll k ” can happen together; use the complement “no six in three rolls”.

5 Ignoring irrelevant rolls

6 marks

- (a) Ignore 3, 4, 5, 6. Among 1 and 2, each is equally likely to come first: $\frac{1}{2}$.
- (b) Ignore 2, 3, 4. The first roll among 1, 5, 6 is equally likely to be each of them, and two of the three work: $\frac{2}{3}$.
- (c) Ignore everything except 1 and 5. The relevant sequence must start 5, 1: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

M1	(a)	1
A1	(a)	1
M1	(b) relevant outcomes $\{1, 5, 6\}$	1
A1	(b)	1
M1	(c) relevant sequence starts 5, 1	1
A1	(c)	1

Why this question Each part could be done with an infinite sum, but ignoring rolls that cannot change the answer turns it into a one-line argument.

6 Comparing numbers of heads

6 marks

- (a) Person 1 gets 0 or 1 head, each with probability $\frac{1}{2}$. Person 2 gets 0, 1, 2 with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. $\mathbb{P} = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}$.
- (b) $\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$.
- (c) $1 - \frac{1}{2} - \frac{3}{8} = \frac{1}{8}$.

M1	(a) distributions of heads	1
M1	(a) splitting on Person 1	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1

Why this question The answer to (a) is exactly $\frac{1}{2}$. This is the simplest case of a general pattern: with one extra toss, the person tossing more often is ahead with probability $\frac{1}{2}$.

7 A biased coin

6 marks

- (a) pq .
- (b) HT or TH : $2pq$.
- (c) $q^{n-1}p$.
- (d) The first n tosses are all tails: q^n .

B1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Common mistake In (d), adding up $\mathbb{P}(N = k)$ for $k > n$. That works, but “ $N > n$ ” simply means “no head in the first n tosses”.

8 First to roll a six

6 marks

- (a) A wins on roll $1, 3, 5, \dots$: $\frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \dots = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$.
- (b) The game ends with probability 1, so $\frac{5}{11}$.
- (c) No six in four rolls: $\left(\frac{5}{6}\right)^4 = \frac{625}{1296}$.

M1	(a) A wins on odd rolls	1
M1	(a) geometric series	1
A1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Going first is worth something: $\frac{6}{11}$ against $\frac{5}{11}$. The ratio is 6 : 5 because B needs one more non-six before each attempt.

9 Three dice

6 marks

- (a) $\frac{6 \cdot 5 \cdot 4}{216} = \frac{5}{9}$.
- (b) The orders of 1, 1, 2: 3 outcomes, so $\frac{3}{216} = \frac{1}{72}$.
- (c) $1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) complement	1
A1	(c)	1

Common mistake In (b), counting only $\{1, 1, 2\}$ once. The dice are different, so $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$ are three outcomes.

10 When the face first changes

7 marks

- (a) HT or TH : $2pq$.
- (b) HHT or TTH : $p^2q + q^2p = pq(p + q) = pq$.
- (c) The first $n - 1$ tosses agree, and toss n differs: $p^{n-1}q + q^{n-1}p$.
- (d) $2\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n-1}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b) simplified	1
M1	(c) both cases	1
A1	(c)	1
B1	(d)	1

Why this question Splitting on what happens on toss 1 gives two simple cases. Specialising to $n = 2, 3$ first makes the general formula easy to see.

11 Exam style: two in a row

8 marks

- (a) The second toss matches the first: $\frac{1}{2}$.
- (b) The first toss can be anything. For $N = n$, tosses $2, \dots, n - 1$ must each **differ from the one before** (probability $\frac{1}{2}$ each, $n - 2$ times), and toss n must **match** toss $n - 1$ (probability $\frac{1}{2}$).
So $\mathbb{P}(N = n) = \left(\frac{1}{2}\right)^{n-2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}$.
- (c) $\sum_{n \geq 2} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2} + \frac{1}{4} + \dots = 1$.
- (d) $\sum_{k \geq 1} \left(\frac{1}{2}\right)^{2k-1} = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$.

B1	(a)	1
M1	(b) first toss free	1
M1	(b) middle tosses alternate	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Common mistake Including the first toss in the product and getting $\left(\frac{1}{2}\right)^n$. The first toss can show either face, so it has probability 1.

12 Exam style: a dice game

8 marks

- (a) Sum 7 in 6 ways, 11 in 2: $\frac{8}{36} = \frac{2}{9}$.
- (b) Sums 2, 3, 12 in $1 + 2 + 1$ ways: $\frac{4}{36} = \frac{1}{9}$.
- (c) $1 - \frac{2}{9} - \frac{1}{9} = \frac{2}{3}$.
- (d) $\sum_{k \geq 0} \left(\frac{2}{3}\right)^k \frac{2}{9} = \frac{\frac{2}{9}}{1 - \frac{2}{3}} = \frac{2}{3}$. (Equivalently, ignore the rolls that continue the game: a roll that ends it is a win with probability $\frac{8}{12}$.)
- (e) Two continues, then an end: $\left(\frac{2}{3}\right)^2 \frac{1}{3} = \frac{4}{27}$.

M1	(a)	1
A1	(a)	1
B1	(b)	1
B1	(c)	1
M1	(d) sum over the roll that wins	1
M1	(d) geometric series	1
A1	(d)	1
B1	(e)	1

Why this question The repeated rolls change how long the game lasts, but not who wins: only the roll that ends the game matters.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.