

Topic 13 · Sheet A: Questions

The intermediate value theorem, inverses and attainment of bounds

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 12

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the intermediate value theorem, inverses of strictly monotone continuous functions (including k th roots), and boundedness and attainment of bounds on closed intervals, plus Topics 1 to 12. This is the last sheet, so it draws on the whole module.

Notation. IVT: if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and u lies between $f(a)$ and $f(b)$, then $f(c) = u$ for some $c \in [a, b]$. Polynomials are continuous.

1

4 marks

Use the intermediate value theorem, or explain why it does not apply.

- (a) Show that $x^3 - 3x + 1 = 0$ has a solution in $(0, 1)$. [1]
- (b) Show that $x^5 + x = 3$ has a solution in $(1, 2)$. [1]
- (c) Show that $x^4 = 5$ has a solution in $(1, 2)$. [1]
- (d) $f(x) = \frac{1}{x}$ has $f(-1) = -1$ and $f(1) = 1$. Does the IVT give a $c \in [-1, 1]$ with $f(c) = 0$? [1]

2

4 marks

Let $f(x) = x^2 - 2$ on $[1, 2]$. Starting from $[1, 2]$, halve the interval four times, each time keeping the half where f changes sign. Give the interval after each step.

- (a) Step 1 [1]
- (b) Step 2 [1]
- (c) Step 3 [1]
- (d) Step 4. What number are you approximating? [1]

3

6 marks

- (a) Explain why $f(x) = x^3$ takes the value 5 on $[1, 2]$, and find that point. [2]
- (b) Explain why $f(x) = x^2 - x$ takes the value 4 on $[0, 3]$, and find the point. [2]
- (c) Show that $x^3 = x + 1$ has a solution in $[1, 2]$. [2]

4

6 marks

Let $p(x) = x^3 - 6x^2 + 9x - 1$.

- (a) Compute $p(0), p(1), p(2), p(3), p(4)$, and use the IVT to locate three roots in intervals of length 1. [4]
- (b) Explain why p has exactly three real roots. [2]

5

6 marks

A continuous, strictly monotonic function from an interval onto an interval has a continuous, strictly monotonic inverse.

- (a) Explain why $f(x) = x^3 + 1$ on $\mathbb{R}$ has a continuous inverse, and find it. [2]
- (b) Find the inverse of $f(x) = 2x + 3$ from $(0, 1)$ to $(3, 5)$. [2]
- (c) Find the inverse of $f(x) = x^2$ from $(0, \infty)$ to $(0, \infty)$. Why does x^2 on $(-1, 1)$ have no inverse? [2]

6

6 marks

Use the fact that $x \mapsto x^{1/k}$, the inverse of $x \mapsto x^k$ on $[0, \infty)$, exists and is continuous.

- (a) Explain why there is a unique $y \geq 0$ with $y^3 = 5$. [2]
- (b) Use continuity of $x \mapsto x^{1/3}$ to find $\lim_n (8 + \frac{1}{n})^{1/3}$. [2]
- (c) Find $\lim_n \frac{(n^3 + n)^{1/3}}{n}$. [2]

7

6 marks

- (a) $f(x) = \frac{1}{x}$ on $(0, 1]$ is continuous but unbounded. Why does the boundedness theorem for closed intervals not apply? [2]
- (b) $f(x) = x$ on $(0, 1)$ is bounded. Does it attain its supremum? Why does this not contradict the attainment theorem? [2]
- (c) Find the maximum and minimum of $f(x) = x^2 - 2x$ on $[0, 3]$, and where they are attained. [2]

8

6 marks

- (a) Let $f(x) = x^3 - 3x$ on $[0, 2]$. Use $x^3 - 3x + 2 = (x - 1)^2(x + 2)$ and $x^3 - 3x - 2 = (x + 1)^2(x - 2)$ to find the minimum and maximum of f , and where they are attained. [3]
- (b) Let $f(x) = \frac{1}{1+x^2}$ on $\mathbb{R}$. Find $\sup f$ and $\inf f$. Which is attained? Why does this not contradict the attainment theorem? [3]

9

6 marks

Let $f(x) = \frac{x^2+1}{3}$ on $[0, 1]$, and $g(x) = f(x) - x$.

- (a) Show that f maps $[0, 1]$ into $[0, 1]$. [2]

- (b) Use the IVT on g to show there is a $c \in [0, 1]$ with $f(c) = c$. [2]
- (c) Find c exactly. [2]

10

7 marks

Let $f(x) = x^5 + x$ on $\mathbb{R}$.

- (a) Explain why f is strictly increasing. [2]
- (b) Explain why the image of f is all of $\mathbb{R}$. [2]
- (c) Deduce that f has a continuous inverse, and find $f^{-1}(0)$, $f^{-1}(2)$ and $f^{-1}(34)$. [2]
- (d) Compute $f(0.7)$ and $f(0.8)$, and locate $f^{-1}(1)$ to one decimal place. [1]

11

8 marks

Exam style. Let $p(x) = x^4 - 4x + 1$.

- (a) Compute $p(0)$, $p(1)$ and $p(2)$, and locate two roots. [2]
- (b) Show that $p(x) > 0$ for all $x \leq 0$. [2]
- (c) Compute $p(0.5)$ and $p(0.25)$, and locate the smaller root in an interval of length 0.25. [2]
- (d) p attains a maximum and a minimum on $[0, 2]$. Find the maximum, and say whether the minimum is positive or negative. [2]

12

8 marks

Exam style. Let $f(x) = \frac{x^2 + 1}{x + 1}$ on $[0, 2]$.

- (a) Explain why f is continuous on $[0, 2]$. [1]
- (b) Compute $f(0)$, $f(1)$ and $f(2)$, and show f takes the value 1.2 somewhere in $(1, 2)$. [2]
- (c) Write $f(x) = (x + 1) + \frac{2}{x + 1} - 2$, and use $(x + 1) + \frac{2}{x + 1} - 2\sqrt{2} = \frac{((x + 1) - \sqrt{2})^2}{x + 1}$ to find the minimum of f and where it is attained. [3]
- (d) Find the maximum of f on $[0, 2]$. You may use that f is increasing on $[\sqrt{2} - 1, 2]$. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.