

Topic 13 · Sheet A: Worked answers and mark scheme

The intermediate value theorem, inverses and attainment of bounds

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 12

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Using the IVT

4 marks

(a) $f(0) = 1 > 0 > -1 = f(1)$: **a root in $(0, 1)$** . (b) $f(1) = 2 < 3 < 34 = f(2)$: **a solution in $(1, 2)$** .

(c) $1^4 = 1 < 5 < 16 = 2^4$: **a solution**. (d) **No**: $\frac{1}{x}$ is not defined, let alone continuous, at $0 \in [-1, 1]$.

B1 each part correct

4

Common mistake In (d), applying the IVT anyway. Continuity on the *whole* closed interval is essential.

2 Bisection

4 marks

(a) $f(1.5) = 0.25 > 0$: **$[1, 1.5]$** . (b) $f(1.25) < 0$: **$[1.25, 1.5]$** . (c) $f(1.375) < 0$: **$[1.375, 1.5]$** . (d) $f(1.4375) > 0$: **$[1.375, 1.4375]$** , approximating $\sqrt{2}$.

B1 each step correct

4

Why this question The IVT guarantees a root in each interval, and each step halves the error. This is also the idea behind the proof of Bolzano–Weierstrass.

3 Values other than zero

6 marks

- (a) $f(1) = 1 < 5 < 8 = f(2)$ and f is continuous, so **yes**, at $c = \sqrt[3]{5} \approx 1.71$.
- (b) $f(0) = 0 < 4 < 6 = f(3)$, so **yes**. $x^2 - x - 4 = 0$ gives $c = \frac{1+\sqrt{17}}{2} \approx 2.56$.
- (c) Let $g(x) = x^3 - x - 1$: $g(1) = -1 < 0 < 5 = g(2)$, so **a solution exists**.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b), choosing the root in $[0, 3]$	1
M1	(c) moving everything to one side	1
A1	(c)	1

Why this question To solve $f(x) = g(x)$, apply the IVT to $f - g$ with $u = 0$, as in (c).

4 Counting roots

6 marks

- (a) $-1, 3, 1, -1, 3$. Sign changes give roots in $(0, 1)$, $(2, 3)$ **and** $(3, 4)$.
- (b) These are three different roots, and a cubic has at most three. **Exactly three**.

B1	(a) values	1
M1	(a) using sign changes	1
A2	(a) all three intervals. <i>A1 for two</i>	2
B1	(b) three distinct roots	1
E1	(b) at most three	1

Common mistake Missing the root between 2 and 3. Both $p(1)$ and $p(2)$ are positive, so check every consecutive pair.

5 Inverse functions

6 marks

- (a) Continuous, strictly increasing, with image $\mathbb{R}$. So the inverse exists and is continuous:
 $f^{-1}(y) = \sqrt[3]{y-1}$.
- (b) $f^{-1}(y) = \frac{y-3}{2}$, from $(3, 5)$ to $(0, 1)$.
- (c) $f^{-1}(y) = \sqrt{y}$. On $(-1, 1)$, x^2 is **not injective**: $f(-\frac{1}{2}) = f(\frac{1}{2})$.

E1	(a) hypotheses	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c) inverse	1
E1	(c) not monotone	1

Why this question The inverse function theorem needs strict monotonicity; part (c) shows that without it there may be no inverse.

6 Roots via inverses

6 marks

- (a) $x \mapsto x^3$ is continuous and strictly increasing on $[0, \infty)$ with image $[0, \infty)$. So $5^{1/3}$ **exists and is unique**, as the inverse of x^3 .
- (b) $8 + \frac{1}{n} \rightarrow 8$, so by continuity $(8 + \frac{1}{n})^{1/3} \rightarrow 8^{1/3} = 2$.
- (c) $\frac{(n^3 + n)^{1/3}}{n} = \left(1 + \frac{1}{n^2}\right)^{1/3} \rightarrow 1^{1/3} = 1$.

B1	(a)	1
E1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) taking n inside the root	1
A1	(c)	1

Why this question Continuity lets you move a limit inside a function: $\lim f(x_n) = f(\lim x_n)$. That is the sequential definition at work.

7 Why closed intervals matter

6 marks

- (a) The domain $(0, 1]$ is **not closed**: it is missing 0.
- (b) $\sup f = 1$ is **not attained**. The domain $(0, 1)$ is not closed, so the attainment theorem does not apply.
- (c) $f(x) = (x - 1)^2 - 1$: **minimum -1 at $x = 1$** , and **maximum 3 at $x = 3$** (since $f(0) = 0$).

E1	(a)	1
B1	(a)	1
B1	(b)	1
E1	(b)	1
B1	(c) minimum	1
B1	(c) maximum	1

Why this question These are the two standard examples. They show both theorems fail without a closed interval.

8 Finding extremes

6 marks

- (a) $f(x) + 2 = (x - 1)^2(x + 2) \geq 0$ on $[0, 2]$, with equality at 1: **minimum -2 at $x = 1$** .
 $f(x) - 2 = (x + 1)^2(x - 2) \leq 0$ on $[0, 2]$, with equality at 2: **maximum 2 at $x = 2$** .
- (b) **$\sup f = 1$** , attained at 0. **$\inf f = 0$** , not attained. The domain $\mathbb{R}$ is not a closed bounded interval.

M1	(a) using the factorisations	1
A1	(a) minimum	1
A1	(a) maximum	1
B1	(b) $\sup$ attained	1
B1	(b) $\inf$ not attained	1
E1	(b)	1

Why this question The attainment theorem says the extremes exist; a factorisation like (a) tells you where they are, without any calculus.

9 A fixed point

6 marks

- (a) For $x \in [0, 1]$, $1 \leq x^2 + 1 \leq 2$, so $f(x) \in [\frac{1}{3}, \frac{2}{3}] \subseteq [0, 1]$.
- (b) g is continuous, $g(0) = \frac{1}{3} > 0$ and $g(1) = -\frac{1}{3} < 0$, so **$g(c) = 0$ for some c** , i.e. $f(c) = c$.
- (c) $x^2 - 3x + 1 = 0$ gives **$c = \frac{3-\sqrt{5}}{2} \approx 0.382$** (the other root is outside $[0, 1]$).

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question The same argument shows every continuous $f : [0, 1] \rightarrow [0, 1]$ has a fixed point.

10 A continuous inverse

7 marks

- (a) x^5 and x are both strictly increasing, so their sum is. **Strictly increasing.**
- (b) $f(n) \geq n \rightarrow \infty$ and $f(-n) \leq -n \rightarrow -\infty$, so for any y there are $a < b$ with $f(a) < y < f(b)$.
By the IVT, y is a value of f .
- (c) By the inverse function theorem, f^{-1} **exists and is continuous**. $f^{-1}(0) = 0$, $f^{-1}(2) = 1$,
 $f^{-1}(34) = 2$.
- (d) $f(0.7) \approx 0.868 < 1 < 1.128 \approx f(0.8)$, so $f^{-1}(1) \in (0.7, 0.8)$.

M1	(a)	1
A1	(a)	1
M1	(b) unbounded both ways	1
A1	(b) IVT	1
B1	(c) existence	1
B1	(c) the three values	1
B1	(d)	1

Why this question There is no formula for f^{-1} , yet the theorem guarantees it exists and is continuous, and the IVT lets you approximate it.

11 Exam style: locating roots

8 marks

- (a) $p(0) = 1$, $p(1) = -2$, $p(2) = 9$: roots in $(0, 1)$ **and** $(1, 2)$.
- (b) For $x \leq 0$: $x^4 \geq 0$ and $-4x \geq 0$, so $p(x) \geq 1 > 0$.
- (c) $p(0.5) = -0.9375 < 0$ and $p(0.25) \approx 0.0039 > 0$: **the root lies in $(0.25, 0.5)$** .
- (d) $p(1) = -2 < 0$, so **the minimum is negative**. For the maximum: on $[0, 2]$, $x^3 \leq 8$ so $x^4 \leq 8x$, giving $p(x) - 9 = x^4 - 4x - 8 \leq 4x - 8 \leq 0$. So **the maximum is $p(2) = 9$** .

B1	(a) values	1
A1	(a) intervals	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d) maximum	1
B1	(d) minimum negative	1

Why this question Part (b) rules out negative roots, so with (a) you know where to look for all real roots.

12 Exam style: extremes on a closed interval

8 marks

- (a) A quotient of polynomials with $x + 1 \geq 1 > 0$ on $[0, 2]$.
- (b) $f(0) = 1$, $f(1) = 1$, $f(2) = \frac{5}{3}$. Since $1 < 1.2 < \frac{5}{3}$, the IVT on $[1, 2]$ gives **a point in $(1, 2)$ with $f = 1.2$** .
- (c) $f(x) = (x + 1) + \frac{2}{x+1} - 2 \geq 2\sqrt{2} - 2$, with equality when $x + 1 = \sqrt{2}$. So the **minimum is $2\sqrt{2} - 2 \approx 0.828$, at $x = \sqrt{2} - 1$** .
- (d) On $[0, 1]$, $x^2 \leq x$ gives $f(x) \leq 1$; on $[\sqrt{2} - 1, 2]$, $f \leq f(2)$ since f is increasing there. So the **maximum is $\frac{5}{3}$, at $x = 2$** .

B1	(a)	1
B1	(b) values	1
A1	(b) IVT	1
M1	(c) using the identity	1
A1	(c) minimum value	1
A1	(c) where	1
M1	(d)	1
A1	(d)	1

Why this question The attainment theorem promises the minimum exists; the identity in (c) finds it exactly.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.