

Topic 12 · Sheet A: Questions

Continuity: sequential and ϵ - δ definitions

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 11

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need sequential and ϵ - δ continuity (and that they agree), continuity at endpoints, the algebra and composition of continuous functions, and continuous functions staying positive near a positive value, plus Topics 1 to 11.

Notation. f is continuous at c if $x_n \rightarrow c$ implies $f(x_n) \rightarrow f(c)$; equivalently, for every $\epsilon > 0$ there is $\delta > 0$ with $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

1

4 marks

Decide whether each function $\mathbb{R} \rightarrow \mathbb{R}$ is continuous at 0.

- (a) $f(x) = x^3 - 2x$ [1]
- (b) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$ [1]
- (c) $f(x) = |x|$ [1]
- (d) $f(x) = \frac{1}{1+x^2}$ [1]

2

4 marks

- (a) For the function in 1(b), find $\lim_n f(x_n)$ for $x_n = \frac{1}{n}$ and for $x_n = -\frac{1}{n}$. Compare with $f(0)$. [2]
- (b) For $g(x) = x^2$ and $x_n = 2 + \frac{1}{n}$, find $\lim_n g(x_n)$ and compare with $g(2)$. [2]

3

6 marks

Use a sequence to show each function is not continuous at the given point.

- (a) $f(x) = x + 1$ for $x > 0$, and $f(x) = x - 1$ for $x \leq 0$, at $c = 0$ [2]
- (b) $f(x) = \frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$, at $c = 0$ [2]
- (c) $f(x) = \lfloor x \rfloor$, the largest integer at most x , at $c = 2$ [2]

4

6 marks

For each function and point, find a δ in terms of ϵ that works in the ϵ - δ definition.

- (a) $f(x) = 3x + 1$ at $c = 2$. Give δ when $\epsilon = 0.3$. [2]
 (b) $f(x) = -2x + 5$ at $c = -1$ [2]
 (c) $f(x) = 7$ at any c [2]

5

6 marks

Let $f(x) = x^2$.

- (a) At $c = 1$, write $|x^2 - 1| = |x - 1||x + 1|$. Assuming $\delta \leq 1$, bound $|x + 1|$, and find a δ that works for every ϵ . [3]
 (b) Give δ for $\epsilon = 0.03$, and check it at $x = 1.009$. [1]
 (c) Repeat (a) at a general point c . [2]

6

6 marks

Use the algebra of continuous functions and continuity of polynomials. Give the largest intervals on which each function is continuous.

- (a) $f(x) = \frac{x^2 + 1}{x - 3}$ [2]
 (b) $f(x) = (x^3 - x)(x + 5)$ [1]
 (c) $f(x) = \frac{1}{x^2 - 4x + 5}$ [2]
 (d) Why does the quotient rule for continuous functions not apply to $\frac{1}{x}$ on all of $\mathbb{R}$? [1]

7

6 marks

Use the composition rule for continuous functions. Write each function as a composition, and say why it is continuous on $\mathbb{R}$.

- (a) $h(x) = (x^2 - 1)^5$ [2]
 (b) $h(x) = \frac{1}{1 + (x - 1)^4}$ [2]
 (c) $h(x) = |x^2 - 2|$. First show that $x \mapsto |x|$ is continuous, using $||x_n| - |c|| \leq |x_n - c|$. [2]

8

6 marks

If f is continuous at c and $f(c) > 0$, then $f > 0$ on an interval around c . This question follows the proof, using $\epsilon = \frac{f(c)}{2}$ or the ϵ given.

- (a) Let $f(x) = x^2 - 2$ and $c = 2$, so $f(c) = 2$. Taking $\epsilon = 1$, find the largest δ with $|f(x) - 2| < 1$ whenever $|x - 2| < \delta$. Compare with the largest interval around 2 on which $f > 0$. [3]
 (b) Let $f(x) = 3x - 1$ and $c = 1$. With $\epsilon = \frac{f(c)}{2}$, find δ and the interval (a, b) on which $f > 0$ is guaranteed. [3]

9

6 marks

Let $f(x) = x^2$ for $x \leq 1$ and $f(x) = ax + b$ for $x > 1$.

- (a) Find the condition on a and b for f to be continuous at 1. Use sequences from both sides. [3]
- (b) If also $f(2) = 5$, find a and b . [2]
- (c) For $f(x) = \frac{1}{x}$ on $(0, 1]$, what does continuity at the endpoint 1 mean? [1]

10

7 marks

Let $f(x) = \frac{x^2 - 4}{x - 2}$ for $x \neq 2$, and $f(2) = k$.

- (a) Simplify $f(x)$ for $x \neq 2$. [1]
- (b) Find the value of k that makes f continuous at 2. [2]
- (c) With this k , find a δ for $\epsilon = 0.01$ at $c = 2$. [2]
- (d) With $k = 5$, give a sequence showing f is not continuous at 2. [2]

11

8 marks

Exam style. Let $f(x) = x^3$ and $c = 2$.

- (a) Factorise $x^3 - 8$. Assuming $\delta \leq 1$, bound the second factor, and find a δ in terms of ϵ . [4]
- (b) Give δ for $\epsilon = 0.19$. [1]
- (c) Check your δ at $x = 2.009$. [1]
- (d) Show continuity at 2 with sequences instead, naming the rule you use. [2]

12

8 marks

Exam style. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$.

- (a) Show that f is continuous at 0 by finding δ for each ϵ . [2]
- (b) Use the sequence $x_n = 1 + \frac{\sqrt{2}}{n}$ to show f is not continuous at 1. [3]
- (c) Explain why f is not continuous at any $c \neq 0$, using sequences of rationals and of irrationals tending to c . [2]
- (d) Where is f continuous? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.