

Topic 12 · Sheet A: Worked answers and mark scheme

Continuity: sequential and ϵ - δ definitions

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 11

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Continuous at 0?

4 marks

(a) **Yes:** polynomials are continuous. (b) **No:** $f(\frac{1}{n}) = 1 \not\rightarrow 0 = f(0)$. (c) **Yes:** if $x_n \rightarrow 0$ then $|x_n| \rightarrow 0$. (d) **Yes:** a composition of continuous functions, with $1 + x^2 > 0$.

B1 each part correct

4

Why this question Part (c) has a corner at 0, but a corner is not a break. Continuity is about not lifting the pen.

2 Testing with sequences

4 marks

(a) $f(\frac{1}{n}) = 1 \rightarrow 1$, but $f(-\frac{1}{n}) = 0 \rightarrow 0 = f(0)$. One sequence fails, so f is **not continuous at 0**.

(b) $g(x_n) = 4 + \frac{4}{n} + \frac{1}{n^2} \rightarrow 4 = g(2)$.

B1 (a) both limits

1

E1 (a) one bad sequence is enough

1

M1 (b)

1

A1 (b)

1

Common mistake In (a), thinking $f(-\frac{1}{n}) \rightarrow f(0)$ shows continuity. Sequential continuity needs *every* sequence to work.

3 Discontinuities

6 marks

- (a) $x_n = \frac{1}{n} \rightarrow 0$ but $f(x_n) = 1 + \frac{1}{n} \rightarrow 1 \neq -1 = f(0)$. **Not continuous.**
- (b) $x_n = \frac{1}{n} \rightarrow 0$ but $f(x_n) = n \rightarrow \infty$, which does not converge to 0. **Not continuous.**
- (c) $x_n = 2 - \frac{1}{n} \rightarrow 2$ but $f(x_n) = 1 \rightarrow 1 \neq 2 = f(2)$. **Not continuous.**

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) approaching from the left	1
A1	(c)	1

Why this question In (c), approaching from the right gives $f(x_n) = 2$, which works. The choice of side matters.

4 Linear functions

6 marks

- (a) $|f(x) - f(2)| = 3|x - 2| < \epsilon$ when $|x - 2| < \frac{\epsilon}{3}$: $\delta = \frac{\epsilon}{3}$, and $\delta = 0.1$ for $\epsilon = 0.3$.
- (b) $|f(x) - f(-1)| = 2|x + 1|$: $\delta = \frac{\epsilon}{2}$.
- (c) $|f(x) - f(c)| = 0 < \epsilon$ always: **any** $\delta > 0$ works.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
E1	(c)	1

Why this question For $f(x) = mx + b$, $\delta = \frac{\epsilon}{|m|}$ works everywhere: steeper lines need smaller δ .

5 ϵ - δ for x^2

6 marks

- (a) If $|x - 1| < \delta \leq 1$ then $x \in (0, 2)$, so $|x + 1| < 3$ and $|x^2 - 1| < 3\delta$. $\delta = \min\{1, \frac{\epsilon}{3}\}$ works.
- (b) $\delta = 0.01$. At $x = 1.009$: $x^2 - 1 = 0.018081 < 0.03$.

(c) $|x + c| \leq |x - c| + 2|c| < 1 + 2|c|$, so $\delta = \min\{1, \frac{\epsilon}{2|c|+1}\}$.

M1	(a) bounding $ x + 1 < 3$	1
M1	(a) taking a minimum	1
A1	(a)	1
B1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question This is the ϵ - δ proof that x^2 is continuous. Compare question 12 of Topic 2, which did this with numbers.

6 Algebra of continuous functions

6 marks

- (a) The denominator is non-zero on each of $(-\infty, 3)$ and $(3, \infty)$, so f is continuous on each.
- (b) A polynomial: **continuous on $\mathbb{R}$** .
- (c) $x^2 - 4x + 5 = (x - 2)^2 + 1 \geq 1$, never zero, so **continuous on $\mathbb{R}$** .
- (d) $g(x) = x$ is 0 at $x = 0$, and the theorem needs $g(x) \neq 0$ on the whole interval.

M1	(a)	1
A1	(a)	1
B1	(b)	1
M1	(c) completing the square	1
A1	(c)	1
E1	(d)	1

Common mistake Saying (a) is “continuous on $\mathbb{R}$ except at 3” without saying what happens at 3. It is not even defined there.

7 Compositions

6 marks

- (a) $h = p \circ q$ with $q(x) = x^2 - 1$ and $p(y) = y^5$, both polynomials, so **continuous**.
- (b) $h = g \circ q$ with $q(x) = 1 + (x - 1)^4 \geq 1$ and $g(y) = \frac{1}{y}$, continuous on $(0, \infty)$. **Continuous**.
- (c) If $x_n \rightarrow c$, then $||x_n| - |c|| \leq |x_n - c| \rightarrow 0$, so $|\cdot|$ is continuous. Then $h = |\cdot| \circ (x^2 - 2)$ is **continuous**.

B1	(a) the two functions	1
B1	(a) composition of continuous functions	1
B1	(b) the two functions	1
E1	(b) image of q lies in $(0, \infty)$	1
M1	(c) continuity of $ \cdot $	1
A1	(c)	1

Why this question The composition rule needs the inner function's values to land where the outer one is continuous. That is the point of checking $q(x) \geq 1$ in (b).

8 Staying positive

6 marks

- (a) $|x^2 - 4| < 1$ means $x \in (\sqrt{3}, \sqrt{5})$. The largest δ is $\min\{2 - \sqrt{3}, \sqrt{5} - 2\} = \sqrt{5} - 2 \approx 0.236$. In fact $f > 0$ on $(\sqrt{2}, 4)$ around 2, so the radius could be $2 - \sqrt{2} \approx 0.586$: **the proof's interval is smaller**.
- (b) $\epsilon = 1$: $|3x - 3| < 1$ iff $|x - 1| < \frac{1}{3}$, so $\delta = \frac{1}{3}$ and $f > 0$ on $(\frac{2}{3}, \frac{4}{3})$.

M1	(a) solving $ x^2 - 4 < 1$	1
A1	(a) δ	1
E1	(a) comparison	1
M1	(b)	1
A1	(b) δ	1
A1	(b) interval	1

Why this question The "positive near a positive value" result only promises *some* interval. Seeing how small it can be helps in the proof of the intermediate value theorem.

9 Joining two pieces

6 marks

- (a) From the left, $x_n = 1 - \frac{1}{n}$ gives $f(x_n) = x_n^2 \rightarrow 1 = f(1)$. From the right, $x_n = 1 + \frac{1}{n}$ gives $f(x_n) \rightarrow a + b$. So we need **$a + b = 1$** .
- (b) $2a + b = 5$ and $a + b = 1$ give **$a = 4, b = -3$** .

- (c) Only sequences in $(0, 1]$ count, so we approach 1 from the left: $\frac{1}{x_n} \rightarrow 1$ for $x_n \rightarrow 1$ with $x_n \leq 1$.

M1	(a) left-hand sequence	1
M1	(a) right-hand sequence	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1

Why this question A piecewise function is continuous at the join exactly when the two pieces meet there.

10 Filling a gap

7 marks

- (a) $f(x) = x + 2$ for $x \neq 2$.
 (b) Any $x_n \rightarrow 2$ with $x_n \neq 2$ gives $f(x_n) \rightarrow 4$, so $k = 4$.
 (c) $|f(x) - 4| = |x - 2|$ (also at $x = 2$), so $\delta = 0.01$.
 (d) $x_n = 2 + \frac{1}{n} \rightarrow 2$, but $f(x_n) = 4 + \frac{1}{n} \rightarrow 4 \neq 5$. **Not continuous.**

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question The original formula is undefined at 2, but its values nearby point to the right value to fill in.

11 Exam style: x^3 at 2

8 marks

- (a) $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$. If $|x - 2| < \delta \leq 1$ then $x \in (1, 3)$, so $x^2 + 2x + 4 < 9 + 6 + 4 = 19$. So $|x^3 - 8| < 19\delta$, and $\delta = \min\{1, \frac{\epsilon}{19}\}$ works.
 (b) $\delta = 0.01$.
 (c) $2.009^3 - 8 \approx 0.1085 < 0.19$. **It checks.**
 (d) If $x_n \rightarrow 2$ then $x_n^3 \rightarrow 8$ by the product rule for limits twice, so $f(x_n) \rightarrow f(2)$.

M1	(a) factorising	1
M1	(a) $x \in (1, 3)$ from $\delta \leq 1$	1
A1	(a) bound 19	1
A1	(a) δ	1
B1	(b)	1
B1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question The two definitions of continuity agree. In practice the sequential one is quicker for algebra, and ϵ - δ is better for explicit bounds.

12 Exam style: continuous at exactly one point

8 marks

- (a) $|f(x) - f(0)| = |f(x)| \leq |x|$ for all x , so $\delta = \epsilon$ works.
- (b) x_n is irrational (else $\sqrt{2} = n(x_n - 1)$ would be rational) and $x_n \rightarrow 1$. But $f(x_n) = 0 \rightarrow 0 \neq 1 = f(1)$. **Not continuous at 1.**
- (c) Since rationals and irrationals are dense, there are rationals $r_n \rightarrow c$ and irrationals $s_n \rightarrow c$. Then $f(r_n) = r_n \rightarrow c$ but $f(s_n) = 0 \rightarrow 0$; since $c \neq 0$ these differ, so **f is not continuous at c .**
- (d) **Only at 0.**

M1	(a) $ f(x) \leq x $	1
A1	(a)	1
B1	(b) x_n irrational	1
B1	(b) $x_n \rightarrow 1$	1
A1	(b) conclusion	1
M1	(c) the two sequences	1
A1	(c)	1
B1	(d)	1

Why this question This uses the density results of Topic 7 to build a function that defeats intuition: continuous at one point and nowhere else.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.