

Topic 11 · Sheet A: Questions

The integral test and alternating series

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 10

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the integral bounds, the integral test (including p -series and logarithmic series), the alternating series test and its error estimate, plus Topics 1 to 10. You may use standard integrals.

Notation. For decreasing f and $m \leq n$: $\int_m^{n+1} f(x) dx \leq \sum_{k=m}^n f(k) \leq \int_{m-1}^n f(x) dx$. For (a_n) decreasing and null, $\sum (-1)^{n+1} a_n$ converges, and $|s_n - a| \leq a_{n+1}$.

1

4 marks

Use the p -series result, that $\sum \frac{1}{n^p}$ converges exactly when $p > 1$, to decide whether each series converges.

(a) $\sum \frac{1}{n^3}$ [1]

(b) $\sum \frac{1}{n^{1/3}}$ [1]

(c) $\sum \frac{1}{n^{1.01}}$ [1]

(d) $\sum \frac{1}{\sqrt{n}}$ [1]

2

4 marks

Does the alternating series test show each series converges? If not, decide another way.

(a) $\sum (-1)^{n+1} \frac{1}{\sqrt{n}}$ [1]

(b) $\sum (-1)^{n+1} \frac{n}{n+1}$ [1]

(c) $\sum (-1)^{n+1} \frac{1}{n^2}$ [1]

(d) $\sum (-1)^{n+1} \frac{1}{\ln(n+1)}$ [1]

3

6 marks

Let $f(x) = \frac{1}{x^2}$ on $[1, \infty)$.

- (a) Compute $\int_1^\infty f(x) dx$, and say what the integral test gives. [2]
- (b) Use the upper integral bound to show $\sum_{n=1}^\infty \frac{1}{n^2} \leq 2$. [2]
- (c) Use the lower bound to show $\sum_{n=1}^\infty \frac{1}{n^2} \geq 1$. [2]

4

6 marks

Use the integral test. Check that f is non-negative and decreasing first.

- (a) $\sum \frac{1}{n^2 + 1}$ [3]
- (b) $\sum \frac{n}{n^2 + 1}$ [3]

5

6 marks

Use the fact that $\sum \frac{1}{(n+1)(\ln(n+1))^p}$ converges exactly when $p > 1$, and the shift rule where needed.

- (a) $\sum \frac{1}{(n+1)\ln(n+1)}$ [2]
- (b) $\sum \frac{1}{(n+1)(\ln(n+1))^2}$ [2]
- (c) $\sum_{n=2}^\infty \frac{1}{n \ln n}$ [2]

6

6 marks

Let s_n be the partial sums of $\sum (-1)^{n+1} \frac{1}{n}$, whose sum is $\ln 2 \approx 0.6931$.

- (a) Compute $s_1, \dots, s_4$. [2]
- (b) Use the error estimate $|s_n - a| \leq a_{n+1}$ to bound $|s_4 - \ln 2|$. [1]
- (c) How many terms guarantee an error below 0.001? [2]
- (d) Find the actual error $|s_4 - \ln 2|$. [1]

7

6 marks

Let s_n be the partial sums of $\sum (-1)^{n+1} \frac{1}{n^2}$.

- (a) Compute $s_1, \dots, s_4$. [2]
- (b) Bound the error $|s_4 - a|$, where a is the sum. [1]
- (c) How many terms guarantee an error below 0.0001? [2]
- (d) Does the series converge absolutely? [1]

8

6 marks

Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

- (a) Use the integral bounds with $f(x) = \frac{1}{x}$ to show $\ln(n+1) \leq H_n \leq 1 + \ln n$. [2]
- (b) Give an interval containing H_{100} . [2]
- (c) Find an n that guarantees $H_n \geq 10$, and compare with the bound $s_{2^k} \geq 1 + \frac{k}{2}$ from Topic 9. [2]

9

6 marks

- (a) Consider $f(x) = -x$ on $[1, \infty)$, which is decreasing. Which hypothesis of the integral test fails, and what goes wrong? [2]
- (b) Compute $\int_1^\infty x^{-2} dx$ and $\int_1^\infty x^{-3} dx$ using $\frac{1}{p-1}$, and check by integrating. [2]
- (c) For $p \leq 0$, why does $\sum n^{-p}$ diverge without the integral test? [2]

10

7 marks

Consider $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$, whose sum is $\frac{\pi}{4}$.

- (a) Check the hypotheses of the alternating series test. [2]
- (b) Compute $s_1, \dots, s_5$ to four decimal places. [2]
- (c) Compare the bound a_6 with the actual error $|s_5 - \frac{\pi}{4}|$. [1]
- (d) Is the convergence absolute or conditional? [2]

11

8 marks

Exam style. Let $f(x) = xe^{-x^2}$ on $[1, \infty)$.

- (a) Show that f is non-negative and decreasing on $[1, \infty)$. [2]
- (b) Compute $\int_1^\infty f(x) dx$. [3]
- (c) Deduce that $\sum ne^{-n^2}$ converges, and use the integral bounds to show its sum is at most $\frac{3}{2e}$. [2]
- (d) Give a lower bound for the sum. [1]

12

8 marks

Exam style. Let $a_n = \frac{n}{n^2+1}$, and consider $\sum (-1)^{n+1} a_n$.

- (a) Show that (a_n) is null. [1]
- (b) Show that $a_n - a_{n+1} = \frac{n^2 + n - 1}{(n^2 + 1)((n+1)^2 + 1)}$, and deduce that (a_n) is decreasing. [3]
- (c) Deduce that the series converges. [1]
- (d) Is the convergence absolute or conditional? [2]

(e) Bound the error in using s_{10} for the sum.

[1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.