

Topic 11 · Sheet A: Worked answers and mark scheme

The integral test and alternating series

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 10

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 p -series

4 marks

(a) $p = 3 > 1$: **converges**. (b) $p = \frac{1}{3} \leq 1$: **diverges**. (c) $p = 1.01 > 1$: **converges**. (d) $p = \frac{1}{2}$: **diverges**.

B1 each part correct

4

Why this question Part (c) shows the line is exactly at $p = 1$: any power above 1, however slightly, converges.

2 Checking the alternating test

4 marks

(a) $\frac{1}{\sqrt{n}}$ is decreasing and null: **converges**. (b) Terms do not tend to 0: **diverges**. (c) **Converges**, absolutely in fact. (d) $\frac{1}{\ln(n+1)}$ is decreasing and null: **converges**.

B1 each part correct

4

Common mistake In (b), applying the alternating test without checking $a_n \rightarrow 0$. It fails, and the null-sequence test shows divergence.

3 Integral bounds for $\sum \frac{1}{n^2}$

6 marks

- (a) $\int_1^\infty x^{-2} dx = [-x^{-1}]_1^\infty = 1$, finite, so **the series converges**.
- (b) $\sum_{k=1}^n f(k) = 1 + \sum_{k=2}^n f(k) \leq 1 + \int_1^n x^{-2} dx < 1 + 1 = 2$.
- (c) $\sum_{k=1}^n f(k) \geq \int_1^{n+1} x^{-2} dx = 1 - \frac{1}{n+1} \rightarrow 1$, so the sum is **at least 1**.

M1	(a)	1
A1	(a)	1
M1	(b) upper bound with $m = 2$	1
A1	(b)	1
M1	(c) lower bound with $m = 1$	1
A1	(c)	1

Why this question The true sum is $\frac{\pi^2}{6} \approx 1.645$, inside $[1, 2]$. The integral test gives estimates as well as convergence.

4 The integral test

6 marks

- (a) $f(x) = \frac{1}{1+x^2}$ is positive and decreasing. $\int_1^\infty f = [\arctan x]_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$: **converges**.
- (b) $f(x) = \frac{x}{x^2+1}$ is positive, and $f'(x) = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. $\int_1^\infty f = [\frac{1}{2} \ln(x^2+1)]_1^\infty = \infty$: **diverges**.

B1	(a) hypotheses	1
M1	(a) integral	1
A1	(a)	1
B1	(b) decreasing for $x \geq 1$	1
M1	(b) integral	1
A1	(b)	1

Common mistake Skipping the check that f is decreasing. In (b) it is only decreasing from $x = 1$, which is all the test needs.

5 Logarithmic series

6 marks

- (a) The logarithmic series with $p = 1$: **diverges**.
- (b) The logarithmic series with $p = 2$: **converges**.
- (c) This is the series in (a) shifted by one place, so **it diverges**, by the shift rule.

B1	(a)	1
E1	(a) $p = 1$	1
B1	(b)	1
E1	(b) $p = 2$	1
M1	(c) recognising the shift	1
A1	(c)	1

Why this question $\sum \frac{1}{n \ln n}$ diverges even though its terms are smaller than $\frac{1}{n}$. It sits right on the boundary between convergence and divergence.

6 The alternating harmonic series

6 marks

- (a) 1, 0.5, 0.8333, 0.5833.
 (b) $|s_4 - \ln 2| \leq a_5 = 0.2$.
 (c) Need $\frac{1}{n+1} \leq 0.001$, so $n \geq 999$.
 (d) $|0.5833 - 0.6931| \approx 0.110$, within the bound.

B2	(a) all four. B1 for two or three	2
B1	(b)	1
M1	(c) $a_{n+1} \leq 0.001$	1
A1	(c)	1
B1	(d)	1

Why this question About a thousand terms for three decimal places: conditionally convergent series often converge slowly.

7 A faster alternating series

6 marks

- (a) 1, 0.75, 0.8611, 0.7986.
 (b) $|s_4 - a| \leq a_5 = 0.04$.
 (c) Need $\frac{1}{(n+1)^2} < 0.0001$, so $n + 1 > 100$: $n \geq 100$.
 (d) **Yes:** $\sum \frac{1}{n^2}$ converges.

B2	(a) all four. B1 for two or three	2
B1	(b)	1
M1	(c)	1
A1	(c)	1
B1	(d)	1

Why this question Compare with question 6: squaring the terms makes the error shrink much faster.

8 Estimating the harmonic series

6 marks

- (a) Lower bound with $m = 1$: $H_n \geq \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$. Upper bound with $m = 2$: $H_n = 1 + \sum_{k=2}^n \frac{1}{k} \leq 1 + \int_1^n \frac{dx}{x} = 1 + \ln n$.
- (b) $4.615 \leq H_{100} \leq 5.605$ (in fact $H_{100} \approx 5.187$).
- (c) $\ln(n+1) \geq 10$ when $n+1 \geq e^{10} \approx 22026.5$: $n = 22026$ works. This is far smaller than the $2^{18} = 262144$ from the grouping argument for the harmonic series.

M1	(a) lower bound	1
A1	(a) upper bound	1
B2	(b) both ends. B1 for one	2
M1	(c)	1
A1	(c)	1

Why this question H_n grows like $\ln n$. That explains both the divergence and how slow it is.

9 Hypotheses and limits of the test

6 marks

- (a) $f(x) = -x$ is **not non-negative**. Its integral is $-\infty < \infty$, so part (i) of the test would wrongly suggest convergence, yet $\sum(-n)$ diverges.
- (b) **1 and $\frac{1}{2}$** : $[-x^{-1}]_1^\infty = 1$ and $[-\frac{1}{2}x^{-2}]_1^\infty = \frac{1}{2}$.
- (c) For $p \leq 0$, $n^{-p} \geq 1$, so the terms do not tend to 0: **diverges by the null-sequence test**.

B1	(a) the failing hypothesis	1
E1	(a)	1
B1	(b) both values	1
B1	(b) checked by integrating	1
M1	(c)	1
A1	(c)	1

Why this question The integral test result only treats $p > 0$. Part (c) fills in the rest.

10 A series for π

7 marks

- (a) $a_n = \frac{1}{2n-1}$ is positive, decreasing (the denominator grows) and null.
- (b) **1, 0.6667, 0.8667, 0.7238, 0.8349**.

(c) Bound $a_6 = \frac{1}{11} \approx 0.091$; actual error $|0.8349 - 0.7854| \approx 0.050$.

(d) $\frac{1}{2n-1} \geq \frac{1}{2n}$ and $\sum \frac{1}{2n}$ diverges, so **conditional**.

B1	(a) decreasing	1
B1	(a) null	1
B2	(b) all five. <i>B1 for three or four</i>	2
B1	(c)	1
M1	(d) comparison	1
A1	(d)	1

Why this question This series gives π , but painfully slowly: the error bound $\frac{1}{2n+1}$ shrinks like $\frac{1}{n}$.

11 Exam style: an integral test with an estimate

8 marks

(a) $f(x) \geq 0$ for $x \geq 0$. $f'(x) = (1 - 2x^2)e^{-x^2} < 0$ for $x \geq 1$, so **decreasing**.

(b) Substitute $u = x^2$: $\int_1^\infty xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2}\right]_1^\infty = \frac{1}{2e}$.

(c) Finite, so **converges**. With $m = 2$: $\sum_{n=1}^N f(n) \leq f(1) + \int_1^\infty f = \frac{1}{e} + \frac{1}{2e} = \frac{3}{2e}$.

(d) With $m = 1$: **the sum is at least** $\frac{1}{2e} \approx 0.18$.

B1	(a) non-negative	1
B1	(a) decreasing, via the derivative	1
M1	(b) substitution	1
M1	(b) evaluating at the limits	1
A1	(b)	1
B1	(c) converges	1
A1	(c) the bound	1
B1	(d)	1

Why this question The actual sum is about 0.405, inside $[0.18, 0.55]$. The test gives both convergence and a usable estimate.

12 Exam style: a conditionally convergent series

8 marks

(a) $a_n = \frac{1/n}{1 + 1/n^2} \rightarrow 0$.

(b) $a_n - a_{n+1} = \frac{n((n+1)^2 + 1) - (n+1)(n^2 + 1)}{(n^2 + 1)((n+1)^2 + 1)} = \frac{n^2 + n - 1}{(n^2 + 1)((n+1)^2 + 1)} > 0$ since $n^2 + n - 1 \geq 1$. **Decreasing**.

(c) By the alternating series test, **it converges**.

(d) $\sum \frac{n}{n^2+1}$ diverges (question 4(b)), so **conditional**.

(e) $|s_{10} - a| \leq a_{11} = \frac{11}{122} \approx 0.090$.

B1	(a)	1
M1	(b) common denominator	1
A1	(b) simplified numerator	1
A1	(b) positive	1
B1	(c)	1
B1	(d) conditional	1
E1	(d) reason	1
B1	(e)	1

Why this question This question uses every series tool: the null-sequence check, monotonicity, the alternating test, the integral test and the error estimate.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.