

## Topic 10 · Sheet A: Questions

# Comparison test, absolute convergence and the ratio test

Suggested time

**About 75 minutes**

Total marks

**73**

Questions

**12**

Builds on

**Topics 1 to 9**

**How to use this sheet.** Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the comparison test (for convergence and divergence), absolute and conditional convergence and the ratio test, plus Topics 1 to 9.

**Facts you may use.**  $\sum \frac{1}{n}$  diverges and  $\sum x^n$  converges for  $|x| < 1$  (Topic 9);  $\sum \frac{1}{n^2}$  converges (Topic 8, question 12);  $(1 + \frac{1}{n})^n \rightarrow e$  (Topic 8).

**1**

4 marks

Use the comparison test, for convergence or divergence, to decide whether each series converges.

(a)  $\sum \frac{1}{2^n + 1}$  [1]

(b)  $\sum \frac{n+1}{n^2}$  [1]

(c)  $\sum \frac{\sin^2 n}{3^n}$  [1]

(d)  $\sum \frac{1}{n - \frac{1}{2}}$  [1]

**2**

4 marks

For each series, find  $\ell = \lim_n \frac{a_{n+1}}{a_n}$  and say what the ratio test tells you.

(a)  $\sum \frac{3^n}{n!}$  [1]

(b)  $\sum \frac{n}{2^n}$  [1]

(c)  $\sum \frac{2^n}{n^2}$  [1]

(d)  $\sum \frac{1}{n}$  [1]

**3**

6 marks

Compare with geometric series.

(a) Show that  $\sum \frac{2^n + 5^n}{3^n + 7^n}$  converges. [3]

(b) Show that  $\sum \frac{4^n + n}{3^n}$  diverges. [3]

**4**

6 marks

Decide whether each series converges absolutely.

(a)  $\sum \frac{(-1)^n}{2^n}$  [2]

(b)  $\sum \frac{\cos n}{3^n}$  [2]

(c)  $\sum \frac{(-1)^{n+1}}{n}$ . You may assume it converges. What kind of convergence does it have? [2]

**5**

6 marks

Use the ratio test.

(a)  $\sum \frac{n^3}{3^n}$  [2]

(b)  $\sum \frac{5^n}{n \cdot 4^n}$  [2]

(c)  $\sum \frac{(2n)!}{(n!)^2 5^n}$  [2]

**6**

6 marks

For each series the ratio test gives  $\ell = 1$ . Decide convergence another way.

(a)  $\sum \frac{1}{n(n+1)}$  [2]

(b)  $\sum \frac{n+1}{n+2}$  [2]

(c)  $\sum \frac{1}{\sqrt{n}}$  [2]

**7**

6 marks

The comparison test also bounds the sum.

(a) Show that  $\sum \frac{1}{n^2 + 2n}$  converges with sum at most 1, by comparing with  $\sum \frac{1}{n(n+1)}$ . Find the exact sum by writing  $\frac{1}{n^2 + 2n} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right)$ . [3]

(b) Show that  $\sum \frac{1}{2^n + n}$  converges, with sum less than 1. [3]

8

6 marks

Decide whether each series converges absolutely, converges conditionally, or diverges.

(a)  $\sum (-1)^n \frac{n}{n+1}$  [2]

(b)  $\sum \left(-\frac{2}{3}\right)^n$  [2]

(c)  $\sum \frac{(-1)^n}{n 2^n}$  [2]

9

6 marks

(a) For  $x > 0$ , use the ratio test to show  $\sum \frac{x^n}{n!}$  converges. [2]

(b) For  $x > 0$ , find for which  $x$  the series  $\sum nx^n$  converges. Deal with  $x = 1$  separately. [4]

10

7 marks

Let  $a_n = \frac{n+1}{n 3^n}$ .

(a) Use the ratio test to show  $\sum a_n$  converges. [2]

(b) Show  $a_n \leq \frac{2}{3^n}$ , and use the comparison test to bound the sum. [3]

(c) Give a lower bound for the sum as well. [2]

11

8 marks

*Exam style.* Decide whether each series converges, with full justification.

(a)  $\sum \frac{3n+1}{n 4^n}$  [2]

(b)  $\sum \frac{n^2+1}{n^3+2}$ . Show first that  $\frac{n^2+1}{n^3+2} \geq \frac{1}{3n}$ . [3]

(c)  $\sum \frac{(-1)^n \sin n}{n^2}$ . Does it converge absolutely? [3]

12

8 marks

*Exam style.* Let  $a_n = \frac{n!}{n^n}$ .

(a) Show that  $\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$ . [3]

(b) Find  $\lim_n \frac{a_{n+1}}{a_n}$ , and decide whether  $\sum a_n$  converges. [2]

(c) Does  $\sum (-1)^n a_n$  converge absolutely? [1]

(d) Decide whether  $\sum \frac{2^n n!}{n^n}$  converges. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.