

Topic 10 · Sheet A: Worked answers and mark scheme

Comparison test, absolute convergence and the ratio test

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 9

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Quick comparisons

4 marks

(a) $\leq \frac{1}{2^n}$: **converges**. (b) $\geq \frac{1}{n}$: **diverges**. (c) $\leq \frac{1}{3^n}$: **converges**. (d) $\geq \frac{1}{n}$: **diverges**.

B1 each part, with a valid comparison

4

Common mistake Comparing in the wrong direction, such as showing a series is *smaller* than a divergent one. That proves nothing.

2 Ratio limits

4 marks

(a) $\ell = 0$: **converges**. (b) $\ell = \frac{1}{2}$: **converges**. (c) $\ell = 2$: **diverges**. (d) $\ell = 1$: **no conclusion**.

B1 each part, ℓ and conclusion

4

Why this question In part (d), the ratio test says nothing when $\ell = 1$.

3 Comparing with geometric series

6 marks

- (a) $0 \leq \frac{2^n + 5^n}{3^n + 7^n} \leq \frac{2^n + 5^n}{7^n} = \left(\frac{2}{7}\right)^n + \left(\frac{5}{7}\right)^n$. Both geometric series converge, so by the sum rule and the comparison test **the series converges**.
- (b) $\frac{4^n + n}{3^n} \geq \left(\frac{4}{3}\right)^n \geq 0$, and $\sum \left(\frac{4}{3}\right)^n$ diverges, so **the series diverges** by comparison.

M1	(a) dropping 3^n from the denominator	1
M1	(a) splitting into two geometric series	1
A1	(a)	1
M1	(b) lower bound $\left(\frac{4}{3}\right)^n$	1
M1	(b) comparison for divergence	1
A1	(b)	1

Why this question To show convergence, make the terms bigger; to show divergence, make them smaller.

4 Absolute convergence

6 marks

- (a) $\sum \frac{1}{2^n}$ converges: **absolutely convergent**.
- (b) $\left| \frac{\cos n}{3^n} \right| \leq \frac{1}{3^n}$: **absolutely convergent**, by comparison.
- (c) $\sum \frac{1}{n}$ diverges, so **not absolutely convergent**. It **converges conditionally**.

B1	(a)	1
E1	(a)	1
M1	(b) comparison of $ a_n $	1
A1	(b)	1
B1	(c) not absolute	1
B1	(c) conditional	1

Why this question Absolute convergence always gives convergence. Part (b) is a series with erratic signs that is handled easily this way.

5 The ratio test

6 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{1}{3} \left(1 + \frac{1}{n}\right)^3 \rightarrow \frac{1}{3}$: **converges**.
- (b) $\frac{a_{n+1}}{a_n} = \frac{5}{4} \cdot \frac{n}{n+1} \rightarrow \frac{5}{4} > 1$: **diverges**.

(c) $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{5(n+1)^2} \rightarrow \frac{4}{5}$: **converges**.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) simplifying the factorials	1
A1	(c)	1

Common mistake In (c), writing $\frac{(2n+2)!}{(2n)!} = 2n+2$. Two factors are new: $(2n+2)(2n+1)$.

6 When the ratio test fails

6 marks

- (a) Telescoping (Topic 9) gives sum 1: **converges**.
 (b) Terms $\rightarrow 1 \neq 0$: **diverges** by the null-sequence test.
 (c) $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$: **diverges** by comparison with $\sum \frac{1}{n}$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Three series with $\ell = 1$: one converges and two diverge, for different reasons. Knowing several tests matters.

7 Bounds on sums

6 marks

- (a) $n^2 + 2n \geq n^2 + n$, so $0 \leq \frac{1}{n^2+2n} \leq \frac{1}{n(n+1)}$, whose sum is 1. So it converges with **sum at most** 1. Telescoping: $\frac{1}{2} \left(1 + \frac{1}{2}\right) = \frac{3}{4}$.
 (b) $0 \leq \frac{1}{2^n+n} < \frac{1}{2^n}$, and $\sum \frac{1}{2^n} = 1$. So it converges with **sum less than 1**.

M1	(a) comparison	1
A1	(a) bound 1	1
A1	(a) exact sum $\frac{3}{4}$	1
M1	(b) comparison	1
A1	(b) convergence	1
A1	(b) bound	1

Why this question The comparison test gives $\sum a_n \leq \sum b_n$ as well as convergence, which is often enough for an estimate.

8 Three kinds of behaviour

6 marks

- (a) Terms do not tend to 0: **diverges**.
 (b) $\sum (\frac{2}{3})^n$ converges: **absolutely convergent**.
 (c) $\frac{1}{n2^n} \leq \frac{1}{2^n}$: **absolutely convergent**.

B1	(a) diverges	1
E1	(a) null-sequence test	1
B1	(b)	1
E1	(b)	1
B1	(c)	1
E1	(c)	1

Common mistake In (a), thinking alternating signs help. The null-sequence test comes first.

9 Series with a parameter

6 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \rightarrow 0 < 1$: **converges for every $x > 0$** .
 (b) $\frac{a_{n+1}}{a_n} = \frac{n+1}{n}x \rightarrow x$. So **converges for $0 < x < 1$** and **diverges for $x > 1$** . At $x = 1$ the terms are $n \not\rightarrow 0$: **diverges**.

M1	(a)	1
A1	(a)	1
M1	(b) ratio	1
A1	(b) $x < 1$ converges	1
A1	(b) $x > 1$ diverges	1
B1	(b) $x = 1$	1

Why this question The case $\ell = 1$ always needs its own argument. Here the null-sequence test settles it.

10 Estimating a sum

7 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{(n+2)n}{3(n+1)^2} \rightarrow \frac{1}{3}$: **converges**.

- (b) $\frac{n+1}{n} \leq 2$, so $a_n \leq \frac{2}{3^n}$, and $\sum \frac{2}{3^n} = 2 \cdot \frac{1}{2} = 1$. So **the sum is at most 1**.
- (c) All terms are positive, so the sum is more than $a_1 = \frac{2}{3}$. **It lies in $(\frac{2}{3}, 1]$.**

M1	(a)	1
A1	(a)	1
M1	(b) $\frac{n+1}{n} \leq 2$	1
M1	(b) geometric sum	1
A1	(b)	1
M1	(c) positive terms	1
A1	(c)	1

Why this question Tests tell you a sum exists; comparisons and partial sums tell you roughly what it is.

11 Exam style: choosing a test

8 marks

- (a) $\frac{3n+1}{n4^n} \leq \frac{4}{4^n}$, a convergent geometric series: **converges**. (The ratio test with $\ell = \frac{1}{4}$ also works.)
- (b) $3(n^2+1)n \geq 3n^3 \geq n^3+2$ for $n \geq 1$, so $\frac{n^2+1}{n^3+2} \geq \frac{1}{3n}$. Since $\sum \frac{1}{3n}$ diverges, **the series diverges**.
- (c) $\left| \frac{(-1)^n \sin n}{n^2} \right| \leq \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges. So it **converges absolutely**, and hence converges.

M1	(a)	1
A1	(a)	1
M1	(b) the inequality	1
M1	(b) divergence of $\sum \frac{1}{3n}$	1
A1	(b)	1
M1	(c) bound on $ a_n $	1
A1	(c) absolutely convergent	1
E1	(c) absolute convergence gives convergence	1

Why this question In (b) the ratio test gives $\ell = 1$ and the null-sequence test says nothing, so a comparison is the only tool that works.

12 Exam style: factorials against powers

8 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1} \right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$.
- (b) By the quotient rule, $\ell = \frac{1}{e} \approx 0.37 < 1$: **converges**.

(c) $|(-1)^n a_n| = a_n$, so **yes**.

(d) The ratio is $\frac{2}{(1 + \frac{1}{n})^n} \rightarrow \frac{2}{e} \approx 0.74 < 1$: **converges**.

M1	(a) writing the ratio	1
M1	(a) cancelling $(n+1)!$ against $n!$	1
A1	(a)	1
B1	(b) $\ell = \frac{1}{e}$	1
B1	(b) converges	1
B1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question This links the ratio test to e from Topic 8. With 3^n instead of 2^n , $\ell = \frac{3}{e} > 1$ and the series diverges.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.