

Topic 9 · Sheet A: Questions

Series: geometric, harmonic and the basic rules

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 8

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need partial sums and convergence of series, the geometric and harmonic series, the sum and shift rules, the boundedness criterion and the null-sequence test, plus Topics 1 to 8.

Notation. In this module, sums start at $n = 1$, so the geometric series is $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ for $|x| < 1$. s_n denotes the n th partial sum.

1

4 marks

Find the partial sums asked for.

- (a) $s_1, \dots, s_4$ for $\sum \frac{1}{2^n}$ [1]
- (b) $s_1, \dots, s_4$ for $\sum (-1)^n$ [1]
- (c) s_4 for $\sum n$ [1]
- (d) s_4 for $\sum \frac{1}{n(n+1)}$ [1]

2

4 marks

Use the geometric series formula to find each sum, or say that the series diverges.

- (a) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ [1]
- (b) $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$ [1]
- (c) $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ [1]
- (d) $\sum_{n=1}^{\infty} 2^n$ [1]

3

6 marks

Find each sum, naming the rules you use.

(a) $\sum_{n=1}^{\infty} \frac{3}{4^n}$ [2]

(b) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$ [2]

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+2}$ [2]

4

6 marks

(a) Use $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ to find a formula for the partial sums of $\sum \frac{1}{n(n+1)}$, and hence its sum. [3]

(b) Use the shift rule, or the same method, to find $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$. [3]

5

6 marks

Use the null-sequence test where it applies.

(a) $\sum \frac{n}{n+1}$ [1]

(b) $\sum (-1)^n$ [1]

(c) $\sum \frac{1}{n}$. What does the null-sequence test tell you? Does the series converge? [2]

(d) $\sum \frac{2n^2}{n^2+5}$ [2]

6

6 marks

Let $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

(a) Compute s_1, s_2, s_4 and s_8 , and check that $s_{2^k} \geq 1 + \frac{k}{2}$ in each case. [3]

(b) Using $s_{2^k} \geq 1 + \frac{k}{2}$, find an N with $s_N \geq 6$. [2]

(c) Does $\sum \frac{1}{2^n}$ converge? Use the sum rule. [1]

7

6 marks

Use the boundedness criterion: a series of non-negative terms converges exactly when its partial sums are bounded.

(a) Show that the partial sums of $\sum \frac{1}{2^n}$ are bounded by 1. [2]

(b) Show that $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$, and deduce that $\sum \frac{1}{n!}$ converges with sum at most 2. [3]

(c) The partial sums of $\sum (-1)^n$ are bounded. Why does the criterion not apply? [1]

8

6 marks

- (a) Find $\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{3^n} \right)$. [2]
- (b) Explain why $\sum \left(\frac{1}{2^n} + \frac{1}{n} \right)$ diverges, using the sum rule. [2]
- (c) Find $\sum_{n=1}^{\infty} 5 \left(-\frac{1}{4} \right)^n$. [2]

9

6 marks

Write each recurring decimal as a geometric series, and hence as a fraction.

- (a) $0.777 \dots$ [2]
- (b) $0.2353535 \dots$ [2]
- (c) $0.999 \dots$ [2]

10

7 marks

The partial sums of a series $\sum a_n$ are $s_n = \frac{2n}{n+1}$.

- (a) Find a_1 , a_2 and a_3 . [2]
- (b) Find a formula for a_n for $n \geq 2$, and check it for $n = 1$. [2]
- (c) Find $\sum_{n=1}^{\infty} a_n$. [1]
- (d) Check that $(a_n) \rightarrow 0$, as it must for a convergent series. [1]
- (e) Find $\sum_{n=1}^{\infty} a_{n+3}$. [1]

11

8 marks

Exam style.

- (a) For which x does $\sum_{n=1}^{\infty} \left(\frac{x}{3} \right)^n$ converge? Find its sum in terms of x . [3]
- (b) Find the x for which this sum equals 2. [2]
- (c) For which x does $\sum_{n=1}^{\infty} (x-1)^n$ converge? Find its sum when $x = \frac{1}{2}$. [3]

12

8 marks

Exam style.

- (a) A ball is dropped from 10 m. After each bounce it rises to $\frac{3}{5}$ of its previous height. Write the total distance travelled as 10 plus a series, and find it. [4]
- (b) Find $\sum_{n=1}^{\infty} \frac{2^n - 1}{4^n}$. [2]
- (c) Decide whether $\sum \frac{n^2}{n^2+1}$ converges. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.