

Topic 9 · Sheet A: Worked answers and mark scheme

Series: geometric, harmonic and the basic rules

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 8

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Partial sums

4 marks

(a) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$. (b) $-1, 0, -1, 0$. (c) 10. (d) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$.

B1 each part correct

4

Why this question A series is defined through its partial sums. Writing a few down is always the first step.

2 Geometric series

4 marks

(a) $\frac{1/3}{2/3} = \frac{1}{2}$. (b) $\frac{-1/2}{3/2} = -\frac{1}{3}$. (c) $\frac{2/5}{3/5} = \frac{2}{3}$. (d) **Diverges**, since $|2| \geq 1$.

B1 each part correct

4

Common mistake Using $\frac{1}{1-x}$. That formula is for sums starting at $n = 0$; in this module sums start at $n = 1$.

3 Sum and shift rules

6 marks

- (a) $3 \sum (\frac{1}{4})^n = 3 \cdot \frac{1/4}{3/4} = 1$, by the sum rule.
- (b) $\sum (\frac{1}{3})^n + \sum (\frac{1}{2})^n = \frac{1}{2} + 1 = \frac{3}{2}$, by the sum rule.
- (c) This is $\sum_{n=3}^{\infty} (\frac{1}{2})^n = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

M1	(a)	1
A1	(a)	1
M1	(b) splitting the fraction	1
A1	(b)	1
M1	(c) removing the first two terms	1
A1	(c)	1

Why this question In part (c), shifting changes the sum, but not whether it converges.

4 Telescoping series

6 marks

- (a) $s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$, so the sum is 1.
- (b) This is the series in (a) without its first term $\frac{1}{2}$: $\frac{1}{2}$.

M1	(a) writing out terms	1
A1	(a) $s_n = 1 - \frac{1}{n+1}$	1
A1	(a) sum 1	1
M1	(b) recognising a shift of (a)	1
M1	(b) subtracting the first term	1
A1	(b)	1

Why this question Telescoping is one of the few ways to find a sum exactly. Most tests only decide whether a series converges.

5 The null-sequence test

6 marks

- (a) Terms $\rightarrow 1 \neq 0$: **diverges**.
- (b) Terms do not tend to 0: **diverges**.
- (c) Terms $\rightarrow 0$, so the test **tells you nothing**. The series **diverges**: it is the harmonic series.
- (d) Terms $\rightarrow 2 \neq 0$: **diverges**.

B1	(a)	1
B1	(b)	1
B1	(c) test inconclusive	1
B1	(c) diverges	1
M1	(d) limit of terms	1
A1	(d)	1

Common mistake Using the test the wrong way round: terms tending to 0 does *not* make a series converge.

6 The harmonic series

6 marks

(a) $s_1 = 1, s_2 = 1.5, s_4 \approx 2.083, s_8 \approx 2.718$, against bounds 1, 1.5, 2, 2.5. **All hold.**

(b) $s_{2^{10}} \geq 1 + 5 = 6$, so $N = 1024$ works.

(c) **No:** if it converged, so would $2 \sum \frac{1}{2^n} = \sum \frac{1}{n}$ by the sum rule.

B2	(a) the four sums. B1 for two or three	2
B1	(a) checks	1
M1	(b)	1
A1	(b)	1
B1	(c)	1

Why this question The harmonic series grows like $\frac{k}{2}$ at $n = 2^k$: it diverges, but extremely slowly.

7 Bounded partial sums

6 marks

(a) $s_n = 1 - \frac{1}{2^n} < 1$, and the terms are positive, so **the series converges.**

(b) $n! = 1 \cdot 2 \cdot 3 \cdots n \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$. So $s_n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} < 2$, and **the series converges with sum at most 2.**

(c) Its terms are not all non-negative, which the criterion requires.

M1	(a)	1
A1	(a)	1
M1	(b) comparing $n!$ with 2^{n-1}	1
M1	(b) bounding the partial sums	1
A1	(b)	1
E1	(c)	1

Why this question Part (b) shows $\sum \frac{1}{n!}$ converges without the ratio test. In fact its sum is $e - 1 \approx 1.718$.

8 Combining series

6 marks

(a) $3 \cdot 1 - 2 \cdot \frac{1}{2} = 2.$

(b) If it converged, then $\sum \frac{1}{n} = \sum \left(\frac{1}{2^n} + \frac{1}{n} \right) - \sum \frac{1}{2^n}$ would converge by the sum rule. **It diverges.**

(c) $5 \cdot \frac{-1/4}{5/4} = -1.$

M1	(a)	1
A1	(a)	1
M1	(b) subtracting the convergent part	1
E1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake In (b), adding sums: " $1 + \infty$ ". Argue with the sum rule instead.

9 Recurring decimals

6 marks

(a) $\sum 7 \cdot \left(\frac{1}{10} \right)^n = 7 \cdot \frac{1}{9} = \frac{7}{9}.$

(b) $\frac{1}{5} + \frac{1}{10} \sum 35 \cdot \left(\frac{1}{100} \right)^n = \frac{1}{5} + \frac{35}{990} = \frac{233}{990}.$

(c) $\sum 9 \cdot \left(\frac{1}{10} \right)^n = 9 \cdot \frac{1}{9} = 1.$

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Every recurring decimal is a geometric series, which is why recurring decimals are always rational.

10 From partial sums to terms

7 marks

(a) $s_1 = 1, s_2 = \frac{4}{3}, s_3 = \frac{3}{2},$ so $a_1 = 1, a_2 = \frac{1}{3}, a_3 = \frac{1}{6}.$

(b) $a_n = s_n - s_{n-1} = \frac{2n}{n+1} - \frac{2(n-1)}{n} = \frac{2}{n(n+1)};$ at $n = 1$ this gives $1 = a_1.$

(c) $s_n \rightarrow 2.$

(d) $\frac{2}{n(n+1)} \rightarrow 0$. **Yes.**

(e) $2 - (a_1 + a_2 + a_3) = 2 - \frac{3}{2} = \frac{1}{2}$.

B2	(a) all three. <i>B1 for two</i>	2
M1	(b) $a_n = s_n - s_{n-1}$	1
A1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e)	1

Why this question $a_{n+1} = s_{n+1} - s_n$ is the identity behind “convergent series have null terms”.

11 Exam style: geometric series with a parameter

8 marks

(a) Converges iff $|\frac{x}{3}| < 1$, i.e. $-3 < x < 3$. The sum is $\frac{x/3}{1 - x/3} = \frac{x}{3 - x}$.

(b) $\frac{x}{3 - x} = 2$ gives $x = 2$, which lies in $(-3, 3)$.

(c) Converges iff $|x - 1| < 1$: $x \in (0, 2)$. At $x = \frac{1}{2}$: $\frac{-1/2}{3/2} = -\frac{1}{3}$.

M1	(a) $ \frac{x}{3} < 1$	1
A1	(a) interval	1
A1	(a) sum	1
M1	(b)	1
A1	(b), checking it is in range	1
M1	(c) $ x - 1 < 1$	1
A1	(c) interval	1
A1	(c) sum	1

Why this question Interval questions like (c) are a first glimpse of power series, which come later in analysis.

12 Exam style: series in context

8 marks

(a) After the drop, the ball rises $10(\frac{3}{5})^n$ and falls the same after bounce n . Total $10 + 2\sum_{n=1}^{\infty} 10(\frac{3}{5})^n = 10 + 20 \cdot \frac{3/5}{2/5} = 10 + 30 = 40$ m.

(b) $\sum(\frac{1}{2})^n - \sum(\frac{1}{4})^n = 1 - \frac{1}{3} = \frac{2}{3}$.

(c) Terms $\rightarrow 1 \neq 0$, so **it diverges** by the null-sequence test.

M1	(a) up and down for each bounce	1
M1	(a) the series $2 \sum 10(\frac{3}{5})^n$	1
A1	(a) its sum 30	1
A1	(a) 40 m	1
M1	(b)	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Common mistake In (a), counting each bounce height only once. The ball goes up and comes back down.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.