

Topic 8 · Sheet A: Questions

Monotone sequences, Bolzano–Weierstrass and Cauchy sequences

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 7

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need bounded monotonic sequences (including recursive sequences and $(1 + \frac{1}{n})^n$), the Bolzano–Weierstrass theorem and Cauchy sequences, plus Topics 1 to 7.

Method for recursive sequences. Show the terms stay in an interval, show the sequence is monotonic, conclude it converges, then let $n \rightarrow \infty$ in the recursion and choose the root that fits your bounds.

1

4 marks

Does the theorem that bounded monotonic sequences converge show that each sequence converges?

- (a) $a_n = 1 - \frac{1}{n}$ [1]
- (b) $a_n = n$ [1]
- (c) $a_n = (-1)^n$ [1]
- (d) $a_n = 2 + \frac{1}{n^2}$ [1]

2

4 marks

Let $a_1 = 1$ and $a_{n+1} = \frac{a_n + 2}{2}$.

- (a) Find a_2 , a_3 and a_4 . [2]
- (b) Guess the limit from your terms. [1]
- (c) Assuming the limit a exists, find it from the recursion. [1]

3

6 marks

Let $a_1 = 0$ and $a_{n+1} = \frac{a_n^2 + 2}{3}$.

- (a) Find a_2, a_3 and a_4 , to four decimal places. [1]
- (b) Show that if $0 \leq a_n < 1$, then $0 \leq a_{n+1} < 1$. [2]
- (c) Show that $a_{n+1} - a_n = \frac{(a_n - 1)(a_n - 2)}{3}$, and deduce that (a_n) is increasing. [2]
- (d) Find $\lim_n a_n$. [1]

4

6 marks

Let $a_1 = 10$ and $a_{n+1} = \frac{a_n + 6}{2}$.

- (a) Find a_2, a_3, a_4 . [1]
- (b) Show that if $a_n \geq 6$, then $a_{n+1} \geq 6$. [1]
- (c) Show that (a_n) is decreasing. [2]
- (d) Explain why (a_n) converges, and find its limit. [2]

5

6 marks

Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

- (a) Compute $a_1, \dots, a_5$ to three decimal places. [2]
- (b) You may use that (a_n) is increasing and $a_{2n} < 4$. Using that limits keep non-strict inequalities, give an interval that must contain $e = \lim_n a_n$. [2]
- (c) Compute a_2, a_4 and a_8 , and check that they are below 4. [2]

6

6 marks

For each bounded sequence, find a convergent subsequence and its limit, as Bolzano–Weierstrass promises.

- (a) $a_n = \frac{(-1)^n n}{n+1}$ [2]
- (b) $a_n = \cos(n\pi/2)$ [2]
- (c) Does (n) have a convergent subsequence? Explain. [2]

7

6 marks

This question runs the bisection idea behind Bolzano–Weierstrass. Let $x_n = \frac{1}{2}\left(1 + \frac{(-1)^n}{n}\right)$, which lies in $[0, 1]$.

- (a) Write down $x_1, \dots, x_6$. [1]
- (b) Starting from $[a_1, b_1] = [0, 1]$, and choosing the left half whenever it contains infinitely many terms, find $[a_2, b_2], [a_3, b_3], [a_4, b_4]$ and $[a_5, b_5]$. [3]
- (c) What is the point c the intervals close in on? Give a subsequence converging to c . [2]

8

6 marks

- (a) For $a_n = \frac{1}{n}$ and $\epsilon = 0.01$, find an N such that $|a_n - a_m| < \epsilon$ for all $n, m > N$. [2]
- (b) Show that $((-1)^n)$ is not Cauchy, by finding an ϵ that fails. [2]
- (c) Let $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$. Show that $s_{2n} - s_n \geq \frac{1}{2}$, and deduce that (s_n) is not Cauchy. [2]

9

6 marks

Let $s_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$.

- (a) For $m > n$, show that $|s_m - s_n| < \frac{1}{2^n}$. Find an N that works in the Cauchy definition for $\epsilon = 0.001$. [3]
- (b) What does “Cauchy implies convergent” tell you? [1]
- (c) Find $\lim_n s_n$. [2]

10

7 marks

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$.

- (a) Show that if $1 \leq a_n < 2$, then $1 \leq a_{n+1} < 2$. [2]
- (b) Show that $a_{n+1}^2 - a_n^2 = (2 - a_n)(1 + a_n)$, and deduce that (a_n) is increasing. [3]
- (c) Find $\lim_n a_n$. [2]

11

8 marks

Exam style. Let $a_1 = 3$ and $a_{n+1} = \frac{a_n^2 + 4}{5}$.

- (a) Find a_2 and a_3 . [1]
- (b) Show that if $1 < a_n \leq 3$, then $1 < a_{n+1} \leq 3$. [2]
- (c) Show that (a_n) is decreasing. [3]
- (d) Find $\lim_n a_n$, explaining which root of the limit equation you choose. [2]

12

8 marks

Exam style.

- (a) Decide whether each sequence must have a convergent subsequence: $(\sin n)$ and $((-1)^n n)$. [2]
- (b) Let $s_n = \sum_{k=1}^n \frac{1}{k^2}$. Explain why (s_n) is increasing. [1]
- (c) Use $\frac{1}{k^2} \leq \frac{1}{k-1} - \frac{1}{k}$ for $k \geq 2$ to show $s_n < 2$ for all n . [3]
- (d) Deduce that (s_n) converges, and say whether it is Cauchy. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.