

Topic 8 · Sheet A: Worked answers and mark scheme

Monotone sequences, Bolzano–Weierstrass and Cauchy sequences

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 7

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 When does the theorem apply?

4 marks

(a) **Yes**: bounded and increasing. (b) **No**: increasing but not bounded (it diverges). (c) **No**: bounded but not monotonic (it diverges). (d) **Yes**: bounded and decreasing.

B1 each part correct

4

Why this question Both hypotheses matter: (b) and (c) each have one and fail to converge.

2 A first recursion

4 marks

(a) $a_2 = 1.5$, $a_3 = 1.75$, $a_4 = 1.875$. (b) 2. (c) $a = \frac{a+2}{2}$ gives $a = 2$.

B2	(a) all three. B1 for two	2
B1	(b)	1
B1	(c)	1

Common mistake Using the limit equation before knowing the limit exists. It only tells you what the limit must be *if* there is one.

3 A recursive sequence, step by step

6 marks

(a) $a_2 \approx 0.6667$, $a_3 \approx 0.8148$, $a_4 \approx 0.8880$.

(b) $a_n^2 \geq 0$ gives $a_{n+1} \geq \frac{2}{3} \geq 0$; and $a_n^2 < 1$ gives $a_{n+1} < \frac{1+2}{3} = 1$.

(c) $a_{n+1} - a_n = \frac{a_n^2 - 3a_n + 2}{3} = \frac{(a_n - 1)(a_n - 2)}{3}$. On $[0, 1)$ both factors are negative, so this is **positive**.

(d) The limit a satisfies $a^2 - 3a + 2 = 0$, so $a = 1$ or 2 . Since $a_n < 1$, $a = 1$.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) factorising	1
A1	(c) sign argument	1
B1	(d)	1

Why this question This is the standard method for recursive sequences. Part (b) provides the boundedness and (c) the monotonicity that the monotone convergence theorem needs.

4 A decreasing recursion

6 marks

(a) 8, 7, 6.5.

(b) If $a_n \geq 6$ then $a_{n+1} \geq \frac{6+6}{2} = 6$.

(c) $a_{n+1} - a_n = \frac{6 - a_n}{2} \leq 0$ since $a_n \geq 6$. **Decreasing**.

(d) Bounded below by 6 and above by $a_1 = 10$, and decreasing, so it converges. $a = \frac{a+6}{2}$ gives $a = 6$.

B1	(a)	1
B1	(b)	1
M1	(c) the difference	1
A1	(c)	1
E1	(d) bounded and monotonic, so convergent	1
A1	(d)	1

Why this question A decreasing sequence is automatically bounded above by its first term, so only the lower bound needs work.

5 The number e

6 marks

- (a) 2, 2.25, 2.370, 2.441, 2.488.
- (b) Every $a_n \geq a_1 = 2$, and every $a_n < 4$ (as $a_n \leq a_{2n} < 4$). Limits keep non-strict inequalities, so $2 \leq e \leq 4$.
- (c) $a_2 = 2.25$, $a_4 \approx 2.441$, $a_8 \approx 2.566$, all below 4.

B2	(a) all five. <i>B1 for three or four</i>	2
M1	(b) bounds on a_n	1
A1	(b)	1
B2	(c) all three. <i>B1 for two</i>	2

Why this question The bound 4 is far from $e \approx 2.718$, but it does not need to be sharp; any bound proves convergence.

6 Convergent subsequences

6 marks

- (a) $a_{2k} = \frac{2k}{2k+1} \rightarrow 1$ (or $a_{2k-1} \rightarrow -1$).
- (b) $a_{4k} = \cos(2k\pi) = 1$, so $(a_{4k}) \rightarrow 1$ (or $a_{2k-1} = 0$).
- (c) **No**: any subsequence has $a_{n_i} = n_i \geq i$, so it is unbounded and cannot converge, since convergent sequences are bounded.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c) no	1
E1	(c) $n_i \geq i$, so unbounded	1

Why this question Part (c) shows why Bolzano–Weierstrass needs the word “bounded”.

7 Bisection by hand

6 marks

- (a) $0, \frac{3}{4}, \frac{1}{3}, \frac{5}{8}, \frac{2}{5}, \frac{7}{12}$.
- (b) The odd terms increase towards $\frac{1}{2}$ from below. $[0, \frac{1}{2}]$: all odd terms. $[0, \frac{1}{4}]$ holds only x_1 , so $[\frac{1}{4}, \frac{1}{2}]$. $[\frac{1}{4}, \frac{3}{8}]$ holds only x_3 , so $[\frac{3}{8}, \frac{1}{2}]$. $[\frac{3}{8}, \frac{7}{16}]$ holds only x_5, x_7 , so $[\frac{7}{16}, \frac{1}{2}]$.
- (c) $c = \frac{1}{2}$, with $(x_{2k-1}) \rightarrow \frac{1}{2}$.

B1	(a)	1
M1	(b) testing which half has infinitely many terms	1
A2	(b) all four intervals. <i>A1 for two or three</i>	2
B1	(c) c	1
B1	(c) subsequence	1

Why this question The proof is not examinable, but running it once makes the idea clear: each step halves the interval while keeping infinitely many terms inside.

8 Cauchy or not

6 marks

- (a) For $n, m > 100$, both $\frac{1}{n}$ and $\frac{1}{m}$ lie in $(0, 0.01)$, so $|a_n - a_m| < 0.01$: $N = 100$.
- (b) With $\epsilon = 1$: $|a_n - a_{n+1}| = 2 \geq 1$ for every n . **Not Cauchy**.
- (c) $s_{2n} - s_n = \frac{1}{n+1} + \dots + \frac{1}{2n} \geq n \cdot \frac{1}{2n} = \frac{1}{2}$. So $\epsilon = \frac{1}{2}$ fails for every N : **not Cauchy**, so (s_n) diverges.

M1	(a)	1
A1	(a)	1
B1	(b) a suitable ϵ	1
E1	(b)	1
M1	(c) n terms each at least $\frac{1}{2n}$	1
A1	(c)	1

Why this question Part (c) is a quick proof that the harmonic series diverges, which comes back in Topic 9.

9 A Cauchy sequence of partial sums

6 marks

- (a) $s_m - s_n = \frac{1}{2^{n+1}} + \dots + \frac{1}{2^m} = \frac{1}{2^n} - \frac{1}{2^m} < \frac{1}{2^n}$. $2^n > 1000$ for $n \geq 10$, so $N = 9$.
- (b) (s_n) is Cauchy, so **it converges**.
- (c) $s_n = 1 - \frac{1}{2^n} \rightarrow 1$.

M1	(a) summing the terms	1
A1	(a) the bound	1
A1	(a) $N = 9$	1
B1	(b)	1
M1	(c) closed form for s_n	1
A1	(c)	1

Why this question “Cauchy implies convergent” lets you prove convergence without knowing the limit. Here you can also find the limit, which is a useful check.

10 Nested square roots

7 marks

- (a) If $1 \leq a_n < 2$ then $3 \leq 2 + a_n < 4$, so $\sqrt{3} \leq a_{n+1} < 2$, and $\sqrt{3} \geq 1$.
- (b) $a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2 - a_n)(1 + a_n) > 0$ on $[1, 2)$. Since all terms are positive, $a_{n+1} > a_n$: **increasing**.
- (c) Bounded and increasing, so it converges to some a , with $a^2 = 2 + a$. So $a = 2$ or -1 ; as $a_n \geq 1$, $a = 2$.

M1	(a)	1
A1	(a)	1
M1	(b) difference of squares	1
A1	(b) factorisation	1
E1	(b) positive terms, so squares compare like the terms	1
M1	(c) limit equation	1
A1	(c)	1

Common mistake In (c), taking the limit of $\sqrt{2 + a_n}$ without justification. Squaring first, $a_{n+1}^2 = 2 + a_n$, avoids needing continuity of the square root.

11 Exam style: a decreasing recursion

8 marks

- (a) $a_2 = 2.6$, $a_3 = 2.152$.
- (b) If $1 < a_n \leq 3$ then $1 < a_n^2 \leq 9$, so $1 = \frac{1+4}{5} < a_{n+1} \leq \frac{9+4}{5} = 2.6 \leq 3$.
- (c) $a_{n+1} - a_n = \frac{a_n^2 - 5a_n + 4}{5} = \frac{(a_n - 1)(a_n - 4)}{5}$. On $(1, 3]$ the first factor is positive and the second negative, so this is **negative**: decreasing.
- (d) Decreasing and bounded below by 1, so it converges to a with $a^2 - 5a + 4 = 0$: $a = 1$ or 4 . Since $a_n \leq 3$, $a = 1$.

B1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) the difference	1
A1	(c) factorised	1
A1	(c) sign on $(1, 3]$	1
M1	(d) limit equation	1
A1	(d) choosing 1 with reason	1

Why this question The two fixed points 1 and 4 bracket the start value 3. The sign of $(a - 1)(a - 4)$ decides which way the sequence moves.

12 Exam style: subsequences and a bounded series

8 marks

- (a) $(\sin n)$ is bounded, so **yes**, by Bolzano–Weierstrass. $((-1)^n n)$: **no**; every subsequence is unbounded since $|a_{n_i}| = n_i \geq i$.
- (b) Each new term $\frac{1}{(n+1)^2}$ is positive.
- (c) $s_n \leq 1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 + \left(1 - \frac{1}{n} \right) < 2$.
- (d) Increasing and bounded, so **it converges**, and therefore **it is Cauchy**, since convergent sequences are Cauchy.

B1	(a) $\sin n$	1
B1	(a) $(-1)^n n$	1
B1	(b)	1
M1	(c) using the given bound	1
M1	(c) the sum telescopes	1
A1	(c)	1
B1	(d) converges	1
B1	(d) Cauchy	1

Why this question Part (c) shows $\sum \frac{1}{n^2}$ converges without the integral test, previewing Topic 11.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.