

Topic 7 · Sheet A: Questions

Rational numbers, suprema, infima and completeness

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 6

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need rational and irrational numbers and how densely they sit in $\mathbb{R}$, upper and lower bounds of sets, suprema and infima, and completeness, plus Topics 1 to 6.

Notation. $\sup A$ is the least upper bound of A and $\inf A$ the greatest lower bound. If $\sup A \in A$ it is called $\max A$; if $\inf A \in A$ it is called $\min A$. You may use that $\sqrt{2}$ is irrational.

1

4 marks

Write each rational number as $\frac{p}{q}$ with p, q coprime.

(a) $\frac{18}{24}$ [1]

(b) $-\frac{35}{49}$ [1]

(c) 0.375 [1]

(d) 0.121212... (recurring) [1]

2

4 marks

Decide whether each number is rational or irrational, using the fact that sums, products and reciprocals of rationals are rational.

(a) $\sqrt{2} + 1$ [1]

(b) $\sqrt{8}/\sqrt{2}$ [1]

(c) $\sqrt{2} \cdot \sqrt{2}$ [1]

(d) $1/\sqrt{2}$ [1]

3

6 marks

- (a) Follow the proof that there is a rational between any two reals, with $a = 0.31$ and $b = 0.32$: choose the smallest $q \in \mathbb{N}$ with $\frac{1}{q} < b - a$, then the largest p with $\frac{p}{q} \leq a$, and find $r = \frac{p+1}{q}$. [3]

- (b) Find an irrational number in $(1, 1.1)$ of the form $r + \sqrt{2}$ with r rational. [3]

4

6 marks

Decide whether each set is bounded above, below, or neither. Give the set as an interval or union of intervals, and give bounds where they exist.

- (a) $\{x \in \mathbb{R} : x^2 < 7\}$ [2]
(b) $\{x \in \mathbb{R} : x^3 > 2\}$ [2]
(c) $\{x \in \mathbb{R} : x^2 \geq 9\}$ [1]
(d) $\{x \in \mathbb{R} : \frac{1}{x} > 2\}$ [1]

5

6 marks

Find $\sup A$ and $\inf A$, and say whether each is a maximum or minimum.

- (a) $A = (-1, 3]$ [2]
(b) $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ [2]
(c) $A = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$ [2]

6

6 marks

Find $\sup A$ and $\inf A$.

- (a) $A = \{x \in \mathbb{Q} : x^2 < 2\}$. Does A have a maximum? [2]
(b) $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$ [2]
(c) $A = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\}$ [2]

7

6 marks

- (a) Let $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ and $\epsilon = 0.01$. Find an $a \in A$ with $\sup A - \epsilon < a \leq \sup A$. [2]
(b) Do the same for $A = (0, 1)$ and $\epsilon = 0.001$. [1]
(c) With the conventions $\sup A = \infty$ when A is unbounded above and $\inf \emptyset = \infty$, write down $\sup \mathbb{N}$, $\inf \mathbb{N}$ and $\inf \emptyset$. [2]
(d) Give two upper bounds of $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$, and say which number is the least upper bound. [1]

8

6 marks

You can find $\inf A$ as $-\sup(-A)$, where $-A = \{-a : a \in A\}$.

- (a) For $A = [-3, 5)$, write down $-A$ and $\sup(-A)$, and hence $\inf A$. [2]
(b) Do the same for $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. [2]
(c) Find $\sup A$ and $\inf A$ for $A = \{x \in \mathbb{R} : x^2 < 5\}$. [2]

9

6 marks

Let $A = \{x \in \mathbb{R} : x^2 + x < 6\}$.

- (a) Write A as an interval, and give three different upper bounds. [1]
- (b) Find $\sup A$. [1]
- (c) Show that 1.9 is not an upper bound, by giving an element of A above it. [1]
- (d) If $U < 2$, give an element of A larger than U (you may assume $U > -3$). [1]
- (e) Find $\inf A$, and say whether it is a minimum. [2]

10

7 marks

Let $A = \{\frac{3n+2}{n+1} : n \in \mathbb{N}\}$.

- (a) Write down the three smallest elements of A . [1]
- (b) Write $\frac{3n+2}{n+1} = 3 - \frac{1}{n+1}$, and find $\inf A$. Is it a minimum? [2]
- (c) Find $\sup A$. Is it a maximum? [2]
- (d) For $\epsilon = 0.01$, find an element of A within ϵ of $\sup A$. [2]

11

8 marks

Exam style. Let $A = \{x \in \mathbb{R} : x^2 - 4x - 5 < 0\} \cup \{7\}$.

- (a) Write A as a union of an interval and a single point. [2]
- (b) Find $\sup A$. Does A have a maximum? [2]
- (c) Find $\inf A$. Does A have a minimum? [2]
- (d) Find $\sup (A \cap (-\infty, 6))$. [1]
- (e) Find $\sup (A \cap \mathbb{Q})$. [1]

12

8 marks

Exam style.

- (a) With $a = \sqrt{2}$, $b = 1.42$ and $q = 200$, take the largest p with $\frac{p}{q} \leq a$ and set $r = \frac{p+1}{q}$, to find a rational number between $\sqrt{2}$ and 1.42. (Use $\sqrt{2} \approx 1.41421$.) [3]
- (b) Find an irrational number between $\frac{1}{3}$ and $\frac{1}{2}$ of the form $r + \sqrt{2}$ with r rational. [2]
- (c) Let $A = \{r \in \mathbb{Q} : r < \sqrt{2}\}$. Find $\sup A$, and explain using the density of $\mathbb{Q}$ why A has no maximum. [2]
- (d) Is $\sup\{r \in \mathbb{Q} : r^2 < 2\}$ a rational number? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.