

Topic 7 · Sheet A: Worked answers and mark scheme

Rational numbers, suprema, infima and completeness

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 6

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Lowest terms

4 marks

(a) $\gcd(18, 24) = 6$: $\frac{3}{4}$. (b) $\gcd(35, 49) = 7$: $-\frac{5}{7}$. (c) $\frac{375}{1000} = \frac{3}{8}$. (d) $x = 0.1212\dots$ gives $100x - x = 12$, so $x = \frac{12}{99} = \frac{4}{33}$.

B1 each part correct

4

Why this question Writing in lowest terms is the step that lets the proof that $\sqrt{2}$ is irrational assume p and q are coprime.

2 Rational or irrational

4 marks

(a) **Irrational**: if $\sqrt{2} + 1$ were rational, so would be $(\sqrt{2} + 1) - 1$. (b) $\sqrt{4} = 2$: **rational**. (c) **Rational**: it is 2. (d) **Irrational**: if $\frac{1}{\sqrt{2}}$ were rational, so would be its reciprocal $\sqrt{2}$.

B1 each part, with a reason for each irrational

4

Common mistake Assuming a product of irrationals is irrational. Part (c) is a counterexample.

3 Finding numbers in an interval

6 marks

- (a) $b - a = 0.01$, so $q = 101$. Then $\frac{31}{101} \approx 0.3069 \leq 0.31 < \frac{32}{101}$, so $p = 31$, and $r = \frac{32}{101} \approx 0.3168$, which lies in $(0.31, 0.32)$.
- (b) Find a rational r in $(1 - \sqrt{2}, 1.1 - \sqrt{2}) \approx (-0.414, -0.314)$, say $r = -0.4$. Then $\sqrt{2} - 0.4 \approx 1.0142$ is irrational and lies in $(1, 1.1)$.

B1	(a) q	1
B1	(a) p	1
B1	(a) r , in the interval	1
M1	(b) shifting the interval by $\sqrt{2}$	1
A1	(b) a valid rational r	1
A1	(b) $r + \sqrt{2}$	1

Why this question Running the proofs with numbers shows they are really algorithms: they tell you how to find the numbers, not just that they exist.

4 Bounded sets

6 marks

- (a) $(-\sqrt{7}, \sqrt{7})$: bounded, with $M = \sqrt{7}$, $m = -\sqrt{7}$.
- (b) $(\sqrt[3]{2}, \infty)$: bounded below by $\sqrt[3]{2}$, not above.
- (c) $(-\infty, -3] \cup [3, \infty)$: neither.
- (d) $(0, \frac{1}{2})$: bounded, with $M = \frac{1}{2}$, $m = 0$.

B1	(a) interval	1
B1	(a) bounds	1
B1	(b) interval	1
B1	(b) bounded below only	1
B1	(c)	1
B1	(d)	1

Common mistake In (d), forgetting that negative x give $\frac{1}{x} < 0$. Use the sign method from Topic 1.

5 Suprema and infima

6 marks

- (a) $\sup A = \max A = 3$; $\inf A = -1$, not a minimum.
- (b) $\inf A = \min A = 0$ (at $n = 1$); $\sup A = 1$, not a maximum.
- (c) Elements $0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \dots$ $\sup A = \max A = \frac{3}{2}$ (at $n = 2$); $\inf A = -1$, not attained, since odd terms are $-1 + \frac{1}{n} > -1$ and approach -1 .

B1	(a) sup	1
B1	(a) inf	1
B1	(b) inf	1
B1	(b) sup	1
B1	(c) sup	1
B1	(c) inf	1

Why this question Remember that $\sup A$ and $\inf A$ need not lie in A . All three parts have one bound inside A and one outside.

6 More suprema

6 marks

- (a) $\sup A = \sqrt{2}$, $\inf A = -\sqrt{2}$. **No maximum**, since $\sqrt{2} \notin \mathbb{Q}$.
- (b) $\sup A = 1$, not attained; $\inf A = \min A = \frac{1}{2}$.
- (c) Largest at $n = m = 1$: $\sup A = \max A = \frac{5}{6}$. Both terms can be made small: $\inf A = 0$, not attained.

B1	(a) sup and inf	1
B1	(a) no maximum	1
B1	(b) sup	1
B1	(b) inf	1
B1	(c) sup	1
B1	(c) inf	1

Why this question Part (a) is the heart of completeness: inside $\mathbb{Q}$ this set has no least upper bound, but inside $\mathbb{R}$ it does.

7 Approximating the supremum

6 marks

- (a) $\sup A = 1$. We need $\frac{1}{n} < 0.01$, so $n = 101$: $a = \frac{100}{101}$.
- (b) For example $a = 0.9995$.
- (c) $\sup \mathbb{N} = \infty$, $\inf \mathbb{N} = 1$, $\inf \emptyset = \infty$.
- (d) For example 2 and 5. **The least upper bound is 1.**

M1	(a)	1
A1	(a)	1
B1	(b)	1
B1	(c) $\sup \mathbb{N}$, $\inf \mathbb{N}$	1
B1	(c) $\inf \emptyset$	1
B1	(d)	1

Common mistake Taking $\inf \mathbb{N} = 0$. In MA142, $\mathbb{N} = \{1, 2, 3, \dots\}$.

8 Infima via $-A$

6 marks

- (a) $-A = (-5, 3]$, so $\sup(-A) = 3$ and $\inf A = -3$.
(b) $-A = \{-\frac{1}{n}\}$, with $\sup(-A) = 0$, so $\inf A = 0$.
(c) $A = (-\sqrt{5}, \sqrt{5})$: $\sup A = \sqrt{5}$, $\inf A = -\sqrt{5}$.

B1	(a) $-A$	1
B1	(a) $\inf A$	1
B1	(b) $\sup(-A)$	1
B1	(b) $\inf A$	1
B1	(c) $\sup$	1
B1	(c) $\inf$	1

Why this question Reflecting a set through 0 swaps upper and lower bounds. That is the whole idea behind $\inf A = -\sup(-A)$.

9 The least upper bound up close

6 marks

- (a) $x^2 + x - 6 = (x + 3)(x - 2) < 0$: $A = (-3, 2)$. Upper bounds: for example 2, 3, 10.
(b) $\sup A = 2$.
(c) For example $1.95 \in A$, and $1.95 > 1.9$.
(d) $\frac{U+2}{2}$ lies between U and 2, so it is in A and above U .
(e) $\inf A = -3$, **not a minimum**, since $-3 \notin A$.

B1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e) infimum	1
B1	(e) not a minimum	1

Why this question Part (d) is the “least” in least upper bound in action: nothing below 2 can be an upper bound, because the midpoint always gets past it.

10 A set from a sequence

7 marks

- (a) $\frac{5}{2}, \frac{8}{3}, \frac{11}{4}$.
- (b) $3 - \frac{1}{n+1}$ increases with n , so $\inf A = \min A = \frac{5}{2}$.
- (c) $\sup A = 3$, **not a maximum**: every element is $3 - \frac{1}{n+1} < 3$, and they get arbitrarily close.
- (d) $\frac{1}{n+1} < 0.01$ needs $n \geq 100$: $\frac{302}{101}$ works.

B1	(a)	1
M1	(b) increasing	1
A1	(b)	1
B1	(c) sup	1
E1	(c) not attained	1
M1	(d)	1
A1	(d)	1

Why this question For an increasing sequence, the supremum of its set of values is its limit. This is exactly why bounded increasing sequences converge.

11 Exam style: an interval plus a point

8 marks

- (a) $(x - 5)(x + 1) < 0$: $A = (-1, 5) \cup \{7\}$.
- (b) $\sup A = 7$, and $\max A = 7$ since $7 \in A$.
- (c) $\inf A = -1$, **no minimum**.
- (d) $A \cap (-\infty, 6) = (-1, 5)$, so 5.
- (e) $7 \in \mathbb{Q}$, so 7.

M1	(a) factorising	1
A1	(a)	1
B1	(b) sup	1
B1	(b) max	1
B1	(c) inf	1
B1	(c) no min	1
B1	(d)	1
B1	(e)	1

Common mistake Thinking $\sup A$ must be the end of an interval. Here an isolated point is the supremum.

12 Exam style: rationals, irrationals and suprema

8 marks

- (a) $\frac{1}{200} = 0.005 < 1.42 - \sqrt{2} \approx 0.0058$. The largest p with $\frac{p}{200} \leq \sqrt{2}$ is $p = 282$, since $200\sqrt{2} \approx 282.8$. So $r = \frac{283}{200} = 1.415$.
- (b) A rational in $(\frac{1}{3} - \sqrt{2}, \frac{1}{2} - \sqrt{2}) \approx (-1.081, -0.914)$ is -1 . So $\sqrt{2} - 1 \approx 0.414$ works.
- (c) $\sup A = \sqrt{2}$. For any $r \in A$, the density of $\mathbb{Q}$ gives a rational s with $r < s < \sqrt{2}$, and $s \in A$.
So no element is largest.
- (d) **No**: it is $\sqrt{2}$, which is irrational.

B1	(a) checking $\frac{1}{q} < b - a$	1
M1	(a) finding $p = 282$	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c) sup	1
E1	(c) no maximum	1
B1	(d)	1

Why this question Parts (c) and (d) together show the gap in $\mathbb{Q}$ that completeness of $\mathbb{R}$ fills.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.