

Topic 6 · Sheet A: Questions

Roots, powers and the ratio lemma

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 5

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need Bernoulli's inequality, the limits $x^{1/n} \rightarrow 1$ and $n^{1/n} \rightarrow 1$, geometric sequences and the ratio lemma, plus Topics 1 to 5.

Notation. Bernoulli's inequality: $(1+x)^n \geq 1+nx$ for $x \geq -1$ and $n \in \mathbb{N}$. The ratio lemma: if $a_n \neq 0$ and $|a_{n+1}/a_n| \rightarrow \ell \in [0, 1)$, then $(a_n) \rightarrow 0$.

1

4 marks

Check Bernoulli's inequality, or explain why it does not apply.

- (a) $x = 0.1, n = 5$ [1]
- (b) $x = -0.1, n = 4$ [1]
- (c) $x = -1, n = 3$ [1]
- (d) $x = -4, n = 3$ [1]

2

4 marks

Use the standard results for (x^n) to decide the behaviour of each geometric sequence.

- (a) $((\frac{2}{3})^n)$ [1]
- (b) $((-\frac{1}{2})^n)$ [1]
- (c) $((\frac{5}{4})^n)$ [1]
- (d) $((-3)^n)$ [1]

3

6 marks

Find each limit, naming the results you use.

- (a) $\lim_n 7^{1/n}$ [1]
- (b) $\lim_n (\frac{1}{5})^{1/n}$ [1]
- (c) $\lim_n (3n)^{1/n}$ [2]
- (d) $\lim_n (n^2)^{1/n}$ [2]

4

6 marks

This question follows the proof that $x^{1/n} \rightarrow 1$, with $x = 4$. Let $a_n = 4^{1/n} - 1$.

- (a) Use Bernoulli's inequality to show $0 \leq a_n \leq \frac{3}{n}$. [2]
- (b) Use (a) to find an N with $|4^{1/n} - 1| < 0.01$ for all $n > N$. [2]
- (c) Evaluate a_{300} on a calculator. Is the bound in (a) sharp? [1]
- (d) Find the smallest N that works in (b). [1]

5

6 marks

Use the ratio lemma to find each limit. Give the limit of the ratio each time.

- (a) $a_n = \frac{n}{3^n}$ [2]
- (b) $a_n = \frac{n^2}{2^n}$ [2]
- (c) $a_n = \frac{5^n}{n!}$ [2]

6

6 marks

For each sequence, find $\lim_n a_{n+1}/a_n$ and the limit of (a_n) if it exists. Explain why the ratio lemma does not help.

- (a) $a_n = \frac{1}{n}$ [2]
- (b) $a_n = 1 + \frac{1}{n^2}$ [2]
- (c) $a_n = n$ [2]

7

6 marks

"Polynomials < powers < factorials." Find each limit.

- (a) $\lim_n \frac{n^{10}}{2^n}$ [2]
- (b) $\lim_n \frac{100^n}{n!}$ [2]
- (c) $\lim_n \left(\frac{2^n}{3^n} + \frac{n}{2^n} \right)$ [2]

8

6 marks

Show each sequence tends to infinity by showing that $(1/a_n)$ is null.

- (a) $a_n = \frac{3^n}{n^2}$ [3]
- (b) $a_n = \frac{n!}{5^n}$ [3]

9

6 marks

Let $a_n = \frac{10^n}{n!}$.

- (a) Find $\frac{a_{n+1}}{a_n}$, and the first n from which it is at most $\frac{1}{2}$. [2]
- (b) For which n is $a_{n+1} \geq a_n$? Which terms are the largest? [2]
- (c) Use the full ratio lemma (where the ratio bound only holds eventually) to find $\lim_n a_n$. [2]

10

7 marks

- (a) Use Bernoulli's inequality to show $\left(1 + \frac{1}{n}\right)^n \geq 2$ for all n . [2]
- (b) Use Bernoulli's inequality to show $\left(1 - \frac{1}{2n}\right)^n \geq \frac{1}{2}$ for all n . [2]
- (c) Use $(1.01)^n \geq 1 + 0.01n$ to find an N such that $(1.01)^n > 1000$ for all $n > N$. Is this the smallest such N ? [3]

11

8 marks

Exam style. Let $a_n = \frac{n^3}{3^n}$.

- (a) Show that $\frac{a_{n+1}}{a_n} = \frac{1}{3} \left(1 + \frac{1}{n}\right)^3$. [2]
- (b) Find the limit of the ratio, and deduce $\lim_n a_n$. [2]
- (c) Find the first n from which $\frac{a_{n+1}}{a_n} \leq \frac{1}{2}$. [2]
- (d) Which term of (a_n) is the largest? [2]

12

8 marks

Exam style. Find each limit.

- (a) $\lim_n \frac{2^n + n^5}{3^n}$ [2]
- (b) $\lim_n \frac{n! + 4^n}{n! - 4^n}$ [3]
- (c) $\lim_n \left((3n)^{1/n} + \left(\frac{1}{2}\right)^{1/n} \right)$ [1]
- (d) $\lim_n (5^n + 3^n)^{1/n}$, by finding a sandwich [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.