

Topic 6 · Sheet A: Worked answers and mark scheme

Roots, powers and the ratio lemma

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 5

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Checking Bernoulli

4 marks

(a) $1.1^5 = 1.61051 \geq 1.5$. **Holds.** (b) $0.9^4 = 0.6561 \geq 0.6$. **Holds.** (c) $0^3 = 0 \geq -2$. **Holds.** (d) **Does not apply**, as $x < -1$; indeed $(-3)^3 = -27 < -11 = 1 + 3(-4)$.

B1 each part correct

4

Why this question Part (d) shows the condition $x \geq -1$ really matters. The proof by induction needs $1 + x \geq 0$ when multiplying the inequality by $1 + x$.

2 Geometric sequences

4 marks

(a) $\rightarrow 0$. (b) $\rightarrow 0$. (c) $\rightarrow \infty$. (d) **No limit**, finite or infinite.

B1 each part correct

4

Common mistake In (b), thinking a negative ratio prevents convergence. Only $x \leq -1$ does.

3 Roots

6 marks

- (a) 1, since $x^{1/n} \rightarrow 1$ for every $x > 0$.
- (b) 1, by the same result, which covers $0 < x < 1$ too.
- (c) $3^{1/n} \cdot n^{1/n} \rightarrow 1 \cdot 1 = 1$, using $x^{1/n} \rightarrow 1$, $n^{1/n} \rightarrow 1$ and the product rule.
- (d) $(n^{1/n})^2 \rightarrow 1^2 = 1$, using $n^{1/n} \rightarrow 1$ and the product rule.

B1	(a)	1
B1	(b)	1
M1	(c) splitting the root	1
A1	(c)	1
M1	(d) writing as a square	1
A1	(d)	1

Why this question Splitting $(ab)^{1/n} = a^{1/n}b^{1/n}$ reduces new limits to the two standard ones, $x^{1/n} \rightarrow 1$ and $n^{1/n} \rightarrow 1$.

4 Inside the proof that $x^{1/n} \rightarrow 1$

6 marks

- (a) $4^{1/n} \geq 1$ so $a_n \geq 0$. By Bernoulli, $4 = (1 + a_n)^n \geq 1 + na_n$, so $a_n \leq \frac{3}{n}$.
- (b) $\frac{3}{n} < 0.01$ for $n > 300$: $N = 300$.
- (c) $a_{300} \approx 0.00463$, well under 0.01. **The bound is not sharp.**
- (d) $4^{1/n} < 1.01$ iff $n > \frac{\ln 4}{\ln 1.01} \approx 139.3$: $N = 139$.

M1	(a) applying Bernoulli to $(1 + a_n)^n$	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
B1	(d)	1

Why this question The proof trades accuracy for simplicity: the bound $\frac{3}{n}$ is crude but proves the limit, which is all that is needed.

5 The ratio lemma at work

6 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{n+1}{3n} \rightarrow \frac{1}{3} < 1$, so $(a_n) \rightarrow 0$.
- (b) $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2} \rightarrow \frac{1}{2} < 1$, so $(a_n) \rightarrow 0$.
- (c) $\frac{a_{n+1}}{a_n} = \frac{5}{n+1} \rightarrow 0 < 1$, so $(a_n) \rightarrow 0$.

M1	(a) ratio	1
A1	(a)	1
M1	(b) ratio	1
A1	(b)	1
M1	(c) ratio, cancelling factorials	1
A1	(c)	1

Common mistake In (c), writing $\frac{(n+1)!}{n!}$ as n . It is $n+1$.

6 When the ratio tends to 1

6 marks

- (a) Ratio $\frac{n}{n+1} \rightarrow 1$; $(a_n) \rightarrow 0$.
- (b) Ratio $\rightarrow 1$; $(a_n) \rightarrow 1$.
- (c) Ratio $\frac{n+1}{n} \rightarrow 1$; $(a_n) \rightarrow \infty$.

In each case $\ell = 1$, which is excluded from the ratio lemma, and the three different outcomes show why: **a ratio limit of 1 gives no information**.

B1	each ratio limit	3
B1	each sequence limit	3

Why this question These three examples together explain the condition $\ell < 1$ better than any single counterexample.

7 Polynomials, powers and factorials

6 marks

- (a) Ratio $\frac{1}{2}(1 + \frac{1}{n})^{10} \rightarrow \frac{1}{2}$: 0.
(b) Ratio $\frac{100}{n+1} \rightarrow 0$: 0.
(c) $(\frac{2}{3})^n \rightarrow 0$ and $\frac{n}{2^n} \rightarrow 0$, so by the sum rule: 0.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) both terms null	1
A1	(c)	1

Why this question In (a), n^{10} is far bigger than 2^n for small n (at $n = 10$, 10^{10} against 1024), yet the power wins eventually.

8 Tending to infinity via reciprocals

6 marks

- (a) $\frac{1}{a_n} = \frac{n^2}{3^n}$, with ratio $\frac{1}{3}(1 + \frac{1}{n})^2 \rightarrow \frac{1}{3}$. So $(\frac{1}{a_n}) \rightarrow 0$, and since $a_n > 0$, $(a_n) \rightarrow \infty$.
(b) $\frac{1}{a_n} = \frac{5^n}{n!}$, with ratio $\frac{5}{n+1} \rightarrow 0$. So $(a_n) \rightarrow \infty$.

M1	(a) reciprocal	1
M1	(a) ratio limit $\frac{1}{3}$	1
A1	(a)	1
M1	(b) reciprocal	1
M1	(b) ratio limit 0	1
A1	(b)	1

Why this question For a positive sequence, $(1/a_n) \rightarrow 0$ means $(a_n) \rightarrow \infty$: given C , use $\epsilon = 1/C$. This is the converse of “tending to infinity gives null reciprocals”.

9 Eventually small ratios

6 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{10}{n+1} \leq \frac{1}{2}$ iff $n \geq 19$.
(b) $a_{n+1} \geq a_n$ iff $n \leq 9$, with equality at $n = 9$. So $a_9 = a_{10} \approx 2755.7$ are the largest.
(c) The ratio is at most $\ell = \frac{1}{2} < 1$ eventually, so $(a_n) \rightarrow 0$.

M1	(a) ratio	1
A1	(a)	1
M1	(b) ratio ≥ 1	1
A1	(b) $a_9 = a_{10}$	1
B1	(c) limit 0	1
E1	(c) “eventually” version of the lemma	1

Why this question The terms grow to nearly 2756 before falling to 0. Only the eventual behaviour of the ratio matters, which is why the full ratio lemma says “eventually”.

10 Bernoulli in action

7 marks

- (a) With $x = \frac{1}{n}$: $(1 + \frac{1}{n})^n \geq 1 + n \cdot \frac{1}{n} = 2$.
- (b) With $x = -\frac{1}{2n} \geq -1$: $(1 - \frac{1}{2n})^n \geq 1 - \frac{n}{2n} = \frac{1}{2}$.
- (c) $1 + 0.01n > 1000$ when $n > 99900$, so $N = 99900$ works. It is **far from the smallest**: $1.01^n > 1000$ already for $n \geq 695$, since $\frac{\ln 1000}{\ln 1.01} \approx 694.2$.

M1	(a) $x = \frac{1}{n}$	1
A1	(a)	1
M1	(b) $x = -\frac{1}{2n}$, checking $x \geq -1$	1
A1	(b)	1
M1	(c) solving $1 + 0.01n > 1000$	1
A1	(c) $N = 99900$	1
B1	(c) not the smallest	1

Why this question Parts (a) and (b) are the two Bernoulli estimates used to show $(1 + \frac{1}{n})^n$ converges.

11 Exam style: a polynomial over a power

8 marks

- (a) $\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3} \left(\frac{n+1}{n} \right)^3 = \frac{1}{3} \left(1 + \frac{1}{n} \right)^3$.
- (b) The ratio tends to $\frac{1}{3}$, so $(a_n) \rightarrow 0$ by the ratio lemma.
- (c) $(1 + \frac{1}{n})^3 \leq \frac{3}{2}$ iff $1 + \frac{1}{n} \leq 1.1447$, i.e. $n \geq 6.9$. So **from** $n = 7$.
- (d) The ratio is above 1 only for $n = 1, 2$, so $a_1 < a_2 < a_3$ and the terms then decrease. $a_3 = 1$ is the largest.

M1	(a)	1
A1	(a)	1
B1	(b) ratio limit	1
B1	(b) limit 0	1
M1	(c)	1
A1	(c)	1
M1	(d) ratio compared with 1	1
A1	(d)	1

Why this question The same ratio gives both the limit and the shape of the sequence: where it rises, where it peaks and how fast it falls.

12 Exam style: mixed limits

8 marks

- (a) $(\frac{2}{3})^n + \frac{n^5}{3^n} \rightarrow 0 + 0 = 0$.
- (b) Divide by $n!$: $\frac{1 + 4^n/n!}{1 - 4^n/n!}$. Since $\frac{4^n}{n!} \rightarrow 0$ (ratio lemma), this tends to 1.
- (c) $1 + 1 = 2$.
- (d) $5^n \leq 5^n + 3^n \leq 2 \cdot 5^n$, so $5 \leq (5^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 5 \rightarrow 5$. By the sandwich theorem: 5.

M1	(a)	1
A1	(a)	1
M1	(b) dividing by $n!$	1
M1	(b) $\frac{4^n}{n!} \rightarrow 0$	1
A1	(b)	1
B1	(c)	1
M1	(d) the sandwich	1
A1	(d)	1

Why this question Part (d) shows the largest term wins inside an n th root. It is a neat use of $2^{1/n} \rightarrow 1$.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.