

Topic 5 · Sheet A: Questions

Computing limits: algebra rules, sandwiches, shifts and subsequences

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 4

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the algebra of limits (sum, product and quotient rules), the sandwich theorem, the shift rule and “eventually”, limits keeping non-strict inequalities, and subsequences, plus Topics 1 to 4.

Notation. You may write $\lim_n a_n$ and work in the short form, but name the rule you use when a question asks for it.

1

4 marks

Compute each limit.

(a) $\lim_n \frac{2n+3}{5n-1}$ [1]

(b) $\lim_n \frac{n^2-4}{3n^2+n}$ [1]

(c) $\lim_n \left(5 - \frac{1}{n}\right) \left(2 + \frac{3}{n^2}\right)$ [1]

(d) $\lim_n \frac{n+1}{n^2+1}$ [1]

2

4 marks

Compute each limit.

(a) $\lim_n \frac{4n^3-n}{2n^3+7}$ [1]

(b) $\lim_n \left(\frac{n+1}{n}\right)^2$ [1]

(c) $\lim_n \frac{3 \cdot 2^n + 1}{2^n}$ [1]

(d) $\lim_n \frac{1+2/n}{1-1/n}$ [1]

3

6 marks

Compute each limit. Show the step where you divide top and bottom, and name each rule (sum, product, quotient) that you use.

$$(a) \lim_n \frac{(n^2 + 1)(3n - 2)}{n^3 + 5} \quad [3]$$

$$(b) \lim_n \frac{2n^2 - 5n}{(n + 1)(n - 3)} \quad [3]$$

4

6 marks

Use the sandwich theorem. Give the two outer sequences and their common limit.

$$(a) c_n = \frac{n + \sin n}{n} \quad [2]$$

$$(b) c_n = \frac{2n + \cos n}{n + 1} \quad [2]$$

$$(c) c_n = \frac{n^2 + (-1)^n n}{n^2 + 1} \quad [2]$$

5

6 marks

$$(a) \text{ Let } a_1 = a_2 = a_3 = 100 \text{ and } a_n = \frac{1}{n-3} \text{ for } n \geq 4. \text{ Find } \lim_n a_n, \text{ explaining why the first three terms do not matter.} \quad [2]$$

$$(b) \text{ Let } a_n = \frac{n-5}{n+5}. \text{ Write down the shifted sequence } (a_{n+5}) \text{ and find its limit.} \quad [2]$$

$$(c) \text{ Let } a_n = n \text{ for } n \leq 1000 \text{ and } a_n = \frac{1}{n} \text{ for } n > 1000. \text{ Is } (a_n) \text{ bounded? Does it converge?} \quad [2]$$

6

6 marks

$$(a) \text{ Let } a_n = (-1)^n + \frac{1}{n}. \text{ Find the limits of the subsequences } (a_{2k}) \text{ and } (a_{2k-1}). \quad [2]$$

$$(b) \text{ Use your subsequences to decide whether } (a_n) \text{ converges.} \quad [2]$$

$$(c) \text{ Let } b_n = \cos(n\pi/3). \text{ Find two subsequences with different limits.} \quad [2]$$

7

6 marks

$$(a) \text{ Every term of } \left(\frac{1}{n^2}\right) \text{ is positive. Is its limit positive? What does this show?} \quad [2]$$

$$(b) \text{ Suppose } (a_n) \rightarrow a \text{ and } 2 \leq a_n \leq 5 \text{ for all } n. \text{ What can you say about } a? \quad [1]$$

$$(c) \text{ Give a sequence with } a_n < 1 \text{ for all } n \text{ and limit } 1. \quad [1]$$

$$(d) \text{ Find the limit of } a_n = \frac{2n+1}{n+2}, \text{ and check that it agrees with the fact that } a_n - 1 \geq 0 \text{ for all } n. \quad [2]$$

8

6 marks

- (a) Find $\lim_n \frac{n + 1/n}{2n - 1}$. [2]
- (b) Explain why the quotient rule cannot be applied directly to $\frac{n + 1}{n - 1}$ as written, and then find the limit. [2]
- (c) Find $\lim_n \frac{\sqrt{n} + 1}{\sqrt{n} - 1}$, for $n \geq 2$. [2]

9

6 marks

- (a) Let $n_i = 2i + 1$. Check that $n_i \geq i$ for all i . [1]
- (b) Let $a_n = \frac{n}{n + 1}$. Find the limit of the subsequence (a_{n^2}) . [1]
- (c) Let $a_n = n \sin(n\pi/2)$. Find three subsequences: one tending to ∞ , one to $-\infty$, and one converging to 0. [3]
- (d) Is the sequence in (c) bounded? [1]

10

7 marks

Let $a_n = \frac{n^2 - 10n}{n^2 + 1}$.

- (a) Find $\lim_n a_n$. [2]
- (b) For which n is $a_n \geq 0$? Explain why the limit must still be ≥ 0 . [2]
- (c) Find the smallest N such that $|a_n - 1| < 0.1$ for all $n > N$. [3]

11

8 marks

Exam style. Let $a_n = \frac{3n^2 + 2n \sin n}{n^2 + 5}$.

- (a) Show that $3n^2 - 2n \leq 3n^2 + 2n \sin n \leq 3n^2 + 2n$. [1]
- (b) Find the limits of $\frac{3n^2 - 2n}{n^2 + 5}$ and $\frac{3n^2 + 2n}{n^2 + 5}$, naming the rules you use. [3]
- (c) Deduce $\lim_n a_n$. [1]
- (d) Find $\lim_n \left(a_n^2 + \frac{1}{a_n}\right)$, explaining why $\frac{1}{a_n}$ is allowed. [3]

12

8 marks

Exam style. Let $b_n = \frac{n + (-1)^n n}{n + 1}$.

- (a) Find $b_1, \dots, b_6$, and the limits of (b_{2k}) and (b_{2k-1}) . Does (b_n) converge? [3]
- (b) Let $c_n = \frac{b_n}{n}$. Show that $0 \leq c_n \leq \frac{2}{n}$, and find $\lim_n c_n$. [2]
- (c) Does the shifted sequence (b_{n+3}) converge? Give a reason. [1]
- (d) Find $\lim_k b_{2k}^2$. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.