

Topic 5 · Sheet A: Worked answers and mark scheme

Computing limits: algebra rules, sandwiches, shifts and subsequences

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 4

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Quick limits

4 marks

(a) $\frac{2}{5}$. (b) $\frac{1}{3}$. (c) $5 \cdot 2 = 10$. (d) 0.

B1 each part correct

4

Why this question Divide top and bottom by the highest power in the denominator. In (d) that is n^2 , giving $\frac{1/n+1/n^2}{1+1/n^2} \rightarrow 0$.

2 More quick limits

4 marks

(a) 2. (b) 1. (c) $3 + \frac{1}{2^n} \rightarrow 3$. (d) 1.

B1 each part correct

4

Common mistake In (c), dividing by n instead of 2^n . Divide by the fastest-growing term.

3 Limits in full detail

6 marks

- (a) Divide by n^3 : $\frac{(1 + \frac{1}{n^2})(3 - \frac{2}{n})}{1 + \frac{5}{n^3}}$. By the quotient rule, then the product and sum rules, this tends to $\frac{(1+0)(3-0)}{1+0} = 3$.
- (b) Expand: $(n+1)(n-3) = n^2 - 2n - 3$. Divide by n^2 : $\frac{2 - \frac{5}{n}}{1 - \frac{2}{n} - \frac{3}{n^2}} \rightarrow \frac{2}{1} = 2$.

M1	(a) dividing by n^3	1
M1	(a) quotient, product and sum rules named	1
A1	(a)	1
M1	(b) expanding and dividing by n^2	1
M1	(b) rules named, denominator limit non-zero	1
A1	(b)	1

Why this question The quotient rule needs the bottom limit to be non-zero. Dividing first is what makes the denominator converge to something non-zero.

4 Sandwiches

6 marks

- (a) $1 - \frac{1}{n} \leq c_n \leq 1 + \frac{1}{n}$, both tending to 1.
- (b) $\frac{2n-1}{n+1} \leq c_n \leq \frac{2n+1}{n+1}$, both tending to 2.
- (c) $\frac{n^2-n}{n^2+1} \leq c_n \leq \frac{n^2+n}{n^2+1}$, both tending to 1.

M1	(a) bounds using $ \sin n \leq 1$	1
A1	(a)	1
M1	(b) bounds using $ \cos n \leq 1$	1
A1	(b)	1
M1	(c) bounds using $(-1)^n = \pm 1$	1
A1	(c)	1

Common mistake Using the sandwich theorem with outer limits that differ. Both outer sequences must tend to the *same* limit.

5 Shifting and “eventually”

6 marks

- (a) a_n agrees with $(\frac{1}{n-3})$ from $n = 4$ on, which tends to 0. Changing finitely many terms does not change a limit, so $\lim_n a_n = 0$.

(b) $a_{n+5} = \frac{n}{n+10}$, which tends to 1. By the shift rule, $(a_n) \rightarrow 1$ as well.

(c) **Bounded**, by 1000. **It converges to 0**, since it agrees with $(\frac{1}{n})$ eventually.

B1	(a) limit	1
E1	(a) finitely many terms do not matter	1
B1	(b) shifted sequence	1
B1	(b) limit	1
B1	(c) bounded	1
B1	(c) converges to 0	1

Why this question Finitely many terms never affect a limit. Part (c) shows they can still make the bounds large.

6 Subsequences

6 marks

(a) $a_{2k} = 1 + \frac{1}{2k} \rightarrow 1$, and $a_{2k-1} = -1 + \frac{1}{2k-1} \rightarrow -1$.

(b) If (a_n) converged, every subsequence would have the same limit. **So it diverges.**

(c) $b_{6k} = \cos(2k\pi) = 1$ and $b_{6k-3} = \cos((2k-1)\pi) = -1$. So $(b_{6k}) \rightarrow 1$ **and** $(b_{6k-3}) \rightarrow -1$.

B1	(a) even terms	1
B1	(a) odd terms	1
B1	(b) diverges	1
E1	(b) subsequences share the limit	1
B1	(c) first subsequence	1
B1	(c) second subsequence	1

Why this question Two subsequences with different limits is the standard way to show divergence.

7 Signs of limits

6 marks

- (a) **No**: the limit is 0. This shows “ $a_n > 0$ ” **does not give** “ $a > 0$ ”.
- (b) $2 \leq a \leq 5$, since limits keep non-strict inequalities.
- (c) For example $a_n = 1 - \frac{1}{n}$.
- (d) $\lim_n a_n = 2$. Also $a_n - 1 = \frac{n-1}{n+2} \geq 0$, so the limit of $a_n - 1$ must be ≥ 0 , and indeed it is 1.

B1	(a) limit 0	1
E1	(a) strict inequality not preserved	1
B1	(b)	1
B1	(c)	1
B1	(d) limit	1
B1	(d) check	1

Common mistake Writing $2 < a < 5$ in (b). Limits only keep non-strict inequalities.

8 Using the quotient rule properly

6 marks

- (a) Divide by n : $\frac{1 + \frac{1}{n^2}}{2 - \frac{1}{n}} \rightarrow \frac{1}{2}$.
- (b) The top and bottom both tend to infinity, so neither converges and the quotient rule does not apply. **Divide by n first**: $\frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} \rightarrow 1$.
- (c) Divide by $\sqrt{n}$: $\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \rightarrow 1$.

M1	(a)	1
A1	(a)	1
E1	(b) top and bottom do not converge	1
A1	(b)	1
M1	(c) dividing by $\sqrt{n}$	1
A1	(c)	1

Why this question The algebra of limits only applies to sequences that already converge. Rewriting first is not a formality; it is what makes the rule usable.

9 Choosing subsequences

6 marks

- (a) $2i + 1 \geq i$ since $i + 1 \geq 0$. **It holds.**
- (b) $a_{n^2} = \frac{n^2}{n^2 + 1} \rightarrow 1$, the same limit as (a_n) .
- (c) $a_{4k+1} = 4k + 1 \rightarrow \infty$; $a_{4k+3} = -(4k + 3) \rightarrow -\infty$; $a_{2k} = 0 \rightarrow 0$.
- (d) **No:** a subsequence is unbounded, so the sequence is too.

B1	(a)	1
B1	(b)	1
B1	(c) each subsequence	3
B1	(d)	1

Why this question Since $\sin(n\pi/2)$ cycles through $1, 0, -1, 0$, the sequence splits naturally into the subsequences $4k + 1$, $4k + 3$ and $2k$.

10 Eventually non-negative

7 marks

- (a) Divide by n^2 : $\frac{1 - \frac{10}{n}}{1 + \frac{1}{n^2}} \rightarrow 1$.
- (b) $a_n \geq 0$ exactly when $n \geq 10$. So $a_n \geq 0$ eventually; applying “ $a_n \geq 0$ gives limit ≥ 0 ” to (a_{n+9}) and using the shift rule gives **a limit ≥ 0** .
- (c) $|a_n - 1| = \frac{10n + 1}{n^2 + 1} < 0.1$ iff $n^2 - 100n - 9 > 0$, which first holds at $n = 101$. So **$N = 100$** .

M1	(a)	1
A1	(a)	1
B1	(b) $n \geq 10$	1
E1	(b) shift, then limits keep ≥ 0	1
M1	(c) $ a_n - 1 = \frac{10n+1}{n^2+1}$	1
M1	(c) solving the quadratic inequality	1
A1	(c) $N = 100$	1

Why this question The first nine terms are negative, yet the limit is positive. “Eventually” is the right idea for limits.

11 Exam style: sandwich, then algebra

8 marks

- (a) Since $-1 \leq \sin n \leq 1$ and $2n > 0$: $-2n \leq 2n \sin n \leq 2n$. Add $3n^2$.
- (b) Dividing by n^2 : $\frac{3 \mp \frac{2}{n}}{1 + \frac{5}{n^2}} \rightarrow \frac{3}{1} = 3$ in both cases, by the quotient and sum rules.
- (c) By the sandwich theorem, $\lim_n a_n = 3$.
- (d) $a_n > 0$ for all n (the lower bound is ≥ 0 for $n \geq 1$), so $\frac{1}{a_n}$ is defined, and its limit is $\frac{1}{3}$ by the quotient rule since $3 \neq 0$. So the limit is $9 + \frac{1}{3} = \frac{28}{3}$.

B1	(a)	1
M1	(b) dividing by n^2	1
A1	(b) lower limit 3	1
A1	(b) upper limit 3	1
B1	(c)	1
E1	(d) $a_n \neq 0$ and limit $\neq 0$	1
M1	(d) product and quotient rules	1
A1	(d)	1

Why this question This combines both big tools: the sandwich theorem deals with $\sin n$, then the algebra rules handle the rest.

12 Exam style: subsequences and shifts

8 marks

- (a) $0, \frac{4}{3}, 0, \frac{8}{5}, 0, \frac{12}{7}$. $b_{2k} = \frac{4k}{2k+1} \rightarrow 2$ and $b_{2k-1} = 0 \rightarrow 0$. **It diverges**, since the two subsequence limits differ.
- (b) $0 \leq b_n \leq \frac{2n}{n+1} < 2$, so $0 \leq c_n \leq \frac{2}{n}$. By the sandwich theorem, $\lim_n c_n = 0$.
- (c) **No**: by the shift rule, (b_{n+3}) converges if and only if (b_n) does.
- (d) $b_{2k} \rightarrow 2$, so by the product rule $b_{2k}^2 \rightarrow 4$.

B1	(a) terms	1
B1	(a) both subsequence limits	1
E1	(a) diverges, with reason	1
M1	(b) the bound	1
A1	(b)	1
B1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question A divergent sequence can still produce convergent sequences: dividing by n in (b) or picking a subsequence in (d).

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.