

Topic 4 · Sheet A: Questions

Null-sequences and convergence

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 3

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need null-sequences, reciprocals of sequences tending to infinity, the absolute value rule, the sandwich theorem and algebra rules for null-sequences, the definition of convergence, the link $(a_n) \rightarrow a \iff (a_n - a) \rightarrow 0$, and “convergent implies bounded”, plus Topics 1 to 3. The algebra rules for general limits come in Topic 5.

Notation. $(a_n) \rightarrow a$ means: for each $\epsilon > 0$ there is $N \in \mathbb{N}$ with $|a_n - a| < \epsilon$ for all $n > N$. “The smallest N ” means the smallest $N \in \mathbb{N}$ for which this works.

1

4 marks

For each null-sequence and each ϵ , find the smallest N such that $|a_n| < \epsilon$ for all $n > N$.

(a) $a_n = \frac{1}{n}, \epsilon = 0.01$ [1]

(b) $a_n = \frac{1}{\sqrt{n}}, \epsilon = 0.1$ [1]

(c) $a_n = \frac{(-1)^n}{n^2}, \epsilon = 0.001$ [1]

(d) $a_n = \frac{1}{n+5}, \epsilon = 0.05$ [1]

2

4 marks

Decide whether each sequence is a null-sequence.

(a) $\frac{1}{n^3}$ [1]

(b) $(-1)^n$ [1]

(c) $\frac{1}{\ln(n+1)}$ [1]

(d) $\frac{n}{n+1}$ [1]

3

6 marks

For a general $\epsilon > 0$, find an N (in terms of ϵ) such that $|a_n| < \epsilon$ for all $n > N$.

(a) $a_n = \frac{3}{n}$ [2]

(b) $a_n = \frac{1}{2n+1}$ [2]

(c) $a_n = \frac{1}{n^2}$ [2]

4

6 marks

Use the sandwich theorem for null-sequences to show each sequence is null. Give the sandwich sequence (b_n) each time.

(a) $a_n = \frac{\sin(n^2)}{n}$ [2]

(b) $a_n = \frac{(-1)^n}{n^2+1}$ [2]

(c) $a_n = \sqrt{n^2+1} - n$ [2]

5

6 marks

Use the sum and product rules for null-sequences, and the sandwich theorem where needed, to show each sequence is null.

(a) $\frac{3n+2}{n^2}$ [2]

(b) $\frac{\sin n}{n^{3/2}}$, written as a product of two sequences [2]

(c) $\frac{5+(-1)^n}{n}$ [2]

6

6 marks

(a) Use the fact that if $(a_n) \rightarrow \infty$ then $(1/a_n) \rightarrow 0$ to explain why $\left(\frac{1}{n^2+1}\right)$ is a null-sequence. [1]

(b) For this sequence and $\epsilon = 0.01$, find the smallest N . [2]

(c) Use the absolute value rule to show $\left(\frac{(-1)^n}{\sqrt{n}}\right)$ is null. [1]

(d) Show that $((-1)^n)$ is not null, by finding an ϵ for which no N works. [2]

7

6 marks

Each sequence converges to the given limit. For the given ϵ , simplify $|a_n - a|$ and find the smallest N .

(a) $a_n = \frac{2n+1}{n}$, $a = 2$, $\epsilon = 0.01$ [2]

(b) $a_n = \frac{3n}{n+2}$, $a = 3$, $\epsilon = 0.1$ [2]

(c) $a_n = 1 + \frac{(-1)^n}{n}$, $a = 1$, $\epsilon = 0.05$ [2]

8

6 marks

Use $(a_n) \rightarrow a \iff (a_n - a) \rightarrow 0$: find the limit a , then show $(a_n - a)$ is a null-sequence.

(a) $a_n = \frac{n-1}{n+1}$ [2]

(b) $a_n = \frac{4n^2+1}{2n^2}$ [2]

(c) $a_n = \frac{n+1}{2n}$ [2]

9

6 marks

For a general $\epsilon > 0$, find an N in terms of ϵ that shows each convergence.

(a) $\frac{5n-1}{n+1} \rightarrow 5$ [3]

(b) $\frac{n+3}{2n} \rightarrow \frac{1}{2}$ [3]

10

7 marks

Let $a_n = \frac{7n}{n+4}$, which converges to 7.

(a) Follow the “convergent implies bounded” argument with $\epsilon = 1$: find the smallest N with $|a_n - 7| < 1$ for $n > N$, and then the bound $M = \max\{|a_1|, \dots, |a_N|, 1 + |a|\}$. [4]

(b) Write $a_n = 7 - \frac{28}{n+4}$ and find a smaller bound for $|a_n|$ directly. [2]

(c) Use “convergent implies bounded” to explain why (n) cannot converge. [1]

11

8 marks

Exam style. Let $a_n = \frac{3n^2+n}{n^2+1}$.

(a) Show that $a_n - 3 = \frac{n-3}{n^2+1}$. [2]

(b) Show that $|a_n - 3| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. [3]

(c) Use (b) to give an N that works for $\epsilon = 0.001$. [1]

(d) Find the smallest N that works for $\epsilon = 0.001$. [2]

12

8 marks

Exam style. Let $a_n = \frac{(-1)^n n}{n^2+1}$ and $b_n = \frac{2n+(-1)^n}{n}$.

(a) Show that (a_n) is null, using $|a_n| \leq \frac{1}{n}$. [2]

(b) Show that $(b_n) \rightarrow 2$, and find the smallest N for $\epsilon = 0.01$. [2]

(c) Let $c_n = a_n b_n$. Show that $|b_n| \leq 3$, and deduce that (c_n) is null. [2]

(d) Use $|c_n| \leq \frac{3}{n}$ to give an N for (c_n) with $\epsilon = 0.01$. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.