

Topic 4 · Sheet A: Worked answers and mark scheme

Null-sequences and convergence

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 3

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Finding N for a given ϵ

4 marks

- (a) $\frac{1}{n} < 0.01$ iff $n > 100$: $N = 100$. (b) $\sqrt{n} > 10$ iff $n > 100$: $N = 100$.
(c) $n^2 > 1000$ iff $n \geq 32$: $N = 31$. (d) $n + 5 > 20$ iff $n > 15$: $N = 15$.

B1 each part correct

4

Common mistake In (a), giving $N = 101$. Since we need $n > N$, $N = 100$ already works.

2 Null or not

4 marks

- (a) **Null**. (b) **Not null**: $|a_n| = 1$. (c) **Null**: $\ln(n+1) \rightarrow \infty$, so its reciprocal is null. (d) **Not null**: $a_n \geq \frac{1}{2}$.

B1 each part correct

4

Why this question Part (c) is null, but very slowly.

3 N as a function of ϵ

6 marks

- (a) $\frac{3}{n} < \epsilon$ iff $n > \frac{3}{\epsilon}$: take **any** $N \geq \frac{3}{\epsilon}$.
- (b) $\frac{1}{2n+1} < \epsilon$ iff $n > \frac{1}{2}(\frac{1}{\epsilon} - 1)$: **any** $N \geq \frac{1}{2\epsilon}$ works.
- (c) $\frac{1}{n^2} < \epsilon$ iff $n > \frac{1}{\sqrt{\epsilon}}$: **any** $N \geq \frac{1}{\sqrt{\epsilon}}$.

M1	(a) rearranging	1
A1	(a)	1
M1	(b) rearranging	1
A1	(b)	1
M1	(c) rearranging	1
A1	(c)	1

Why this question The notes say “choose N such that $\frac{1}{N} < \epsilon$ ”. Here you make that choice explicit, which is what an exam asks for.

4 Sandwiches

6 marks

- (a) $|a_n| \leq \frac{1}{n}$, so $b_n = \frac{1}{n}$.
- (b) $|a_n| = \frac{1}{n^2+1} \leq \frac{1}{n^2}$, so $b_n = \frac{1}{n^2}$, which is null by the product rule.
- (c) $\sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1} + n} \leq \frac{1}{2n}$, so $b_n = \frac{1}{2n}$.

M1	(a) $ \sin \leq 1$	1
A1	(a)	1
M1	(b)	1
A1	(b)	1
M1	(c) multiplying by the conjugate	1
A1	(c)	1

Why this question In part (c), the conjugate trick turns a difference of two large numbers into a small fraction.

5 Algebra of null-sequences

6 marks

- (a) $\frac{3n+2}{n^2} = 3 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n^2}$, **null by the sum rule**.
- (b) $\frac{\sin n}{n} \cdot \frac{1}{\sqrt{n}}$: both factors are null, so **null by the product rule**.

(c) $\left| \frac{5 + (-1)^n}{n} \right| \leq \frac{6}{n}$, and $(\frac{6}{n})$ is null, so **null by the sandwich theorem**.

M1	(a) splitting into $\frac{1}{n}$ and $\frac{1}{n^2}$	1
A1	(a)	1
M1	(b) the two null factors	1
A1	(b)	1
M1	(c) the bound $\frac{6}{n}$	1
A1	(c)	1

Why this question Split a fraction into pieces you already know are null.

6 Reciprocals and absolute values

6 marks

(a) $(n^2 + 1) \rightarrow \infty$, so **its reciprocal is null**.

(b) $n^2 + 1 > 100$ iff $n \geq 10$: **$N = 9$** .

(c) $\left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}}$, which is null, so **the sequence is null**.

(d) With $\epsilon = \frac{1}{2}$: $|(-1)^n| = 1 \geq \frac{1}{2}$ for every n , so **no N works**.

B1	(a)	1
M1	(b)	1
A1	(b)	1
B1	(c)	1
B1	(d) a suitable ϵ	1
E1	(d) every term fails	1

Common mistake In (d), choosing $\epsilon = 2$. Any $\epsilon \leq 1$ works, but a large ϵ proves nothing.

7 Convergence for a given ϵ

6 marks

- (a) $|a_n - 2| = \frac{1}{n} < 0.01$ iff $n > 100$: $N = 100$.
(b) $|a_n - 3| = \frac{6}{n+2} < 0.1$ iff $n > 58$: $N = 58$.
(c) $|a_n - 1| = \frac{1}{n} < 0.05$ iff $n > 20$: $N = 20$.

M1	(a) simplifying $ a_n - 2 $	1
A1	(a)	1
M1	(b) $ a_n - 3 = \frac{6}{n+2}$	1
A1	(b)	1
M1	(c)	1
A1	(c)	1

Why this question Simplifying $|a_n - a|$ to a single fraction first makes every ϵ - N question routine.

8 Reducing to null-sequences

6 marks

- (a) $a = 1$: $a_n - 1 = -\frac{2}{n+1}$, which is null.
(b) $a = 2$: $a_n - 2 = \frac{1}{2n^2}$, which is null.
(c) $a = \frac{1}{2}$: $a_n - \frac{1}{2} = \frac{1}{2n}$, which is null.

B1	(a) limit	1
B1	(a) null difference	1
B1	(b) limit	1
B1	(b) null difference	1
B1	(c) limit	1
B1	(c) null difference	1

Why this question Since $(a_n) \rightarrow a$ exactly when $(a_n - a) \rightarrow 0$, every convergence question is really a null-sequence question, so all the null-sequence tools apply.

9 General ϵ

6 marks

- (a) $\left| \frac{5n-1}{n+1} - 5 \right| = \frac{6}{n+1} < \epsilon$ iff $n > \frac{6}{\epsilon} - 1$. So **any** $N \geq \frac{6}{\epsilon}$ works.
(b) $\left| \frac{n+3}{2n} - \frac{1}{2} \right| = \frac{3}{2n} < \epsilon$ iff $n > \frac{3}{2\epsilon}$. So **any** $N \geq \frac{3}{2\epsilon}$ works.

M1	(a) simplifying $ a_n - 5 $	1
M1	(a) solving for n	1
A1	(a)	1
M1	(b) simplifying	1
M1	(b) solving for n	1
A1	(b)	1

Common mistake Writing $N = \frac{6}{\epsilon}$ without comment when it is not a natural number. Say “any natural number $N \geq \frac{6}{\epsilon}$ ”.

10 Convergent sequences are bounded

7 marks

- (a) $|a_n - 7| = \frac{28}{n+4} < 1$ iff $n > 24$: $N = 24$. The sequence is increasing and below 7, so $|a_1|, \dots, |a_{24}|$ are all less than 7, and $1 + |a| = 8$. So $M = 8$.
- (b) $0 < a_n = 7 - \frac{28}{n+4} < 7$, so $|a_n| < 7$.
- (c) (n) is not bounded, so **it cannot converge**, since convergent sequences are bounded.

M1	(a) $ a_n - 7 = \frac{28}{n+4}$	1
A1	(a) $N = 24$	1
M1	(a) comparing the first N terms with $1 + a $	1
A1	(a) $M = 8$	1
M1	(b)	1
A1	(b)	1
E1	(c)	1

Why this question The “convergent implies bounded” argument gives a bound, not the best one. Part (b) shows a direct look can do better.

11 Exam style: an explicit ϵ - N argument

8 marks

- (a) $a_n - 3 = \frac{3n^2 + n - 3n^2 - 3}{n^2 + 1} = \frac{n - 3}{n^2 + 1}$.
- (b) For $n = 1$: $|a_1 - 3| = \frac{2}{2} = 1 \leq 1$. For $n \geq 2$: $|n - 3| \leq n$, so $|a_n - 3| \leq \frac{n}{n^2 + 1} < \frac{n}{n^2} = \frac{1}{n}$.
- (c) $\frac{1}{n} < 0.001$ for $n > 1000$, so $N = 1000$ works.
- (d) $\frac{n - 3}{n^2 + 1} < 0.001$ first holds for all $n > 996$: $N = 996$. (At $n = 996$ the value is just over 0.001.)

M1	(a) common denominator	1
A1	(a)	1
B1	(b) the case $n = 1$	1
M1	(b) $ n - 3 \leq n$ for $n \geq 2$	1
A1	(b)	1
B1	(c)	1
M1	(d) solving or testing near 1000	1
A1	(d) $N = 996$	1

Why this question A clean bound like $\frac{1}{n}$ makes N easy to find. The exact smallest N is harder, and the definition never needs it.

12 Exam style: null times bounded

8 marks

- (a) $|a_n| = \frac{n}{n^2 + 1} \leq \frac{n}{n^2} = \frac{1}{n}$, so (a_n) is null by the sandwich theorem.
- (b) $b_n - 2 = \frac{(-1)^n}{n}$, which is null, so $(b_n) \rightarrow 2$. $\frac{1}{n} < 0.01$ iff $n > 100$: $N = 100$.
- (c) $|b_n| \leq \frac{2n + 1}{n} \leq 3$. So $|c_n| \leq 3|a_n| \leq \frac{3}{n}$, and (c_n) is null by the sum rule and the sandwich theorem.
- (d) $\frac{3}{n} < 0.01$ iff $n > 300$: $N = 300$.

M1	(a)	1
A1	(a)	1
B1	(b) convergence	1
B1	(b) $N = 100$	1
M1	(c) $ b_n \leq 3$	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question A null-sequence times a bounded sequence is always null. The same idea shows $\frac{\cos n}{n}$ is null, with bounded factor $\cos n$.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.