

Topic 3 · Sheet A: Questions

Sequences: monotonic, bounded and tending to infinity

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 2

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need sequences, monotonic sequences, bounded sequences and sequences tending to $\pm\infty$, plus Topics 1 and 2.

Notation. In this module, $\mathbb{N} = \{1, 2, 3, \dots\}$ and every sequence (a_n) starts at $n = 1$. $(a_n) \rightarrow \infty$ means: for every $C > 0$ there is $N \in \mathbb{N}$ with $a_n > C$ whenever $n > N$.

1

4 marks

Write down the first five terms of each sequence.

(a) $a_n = \frac{(-1)^n}{n}$ [1]

(b) $a_n = n^2 - 3n$ [1]

(c) $a_n = \cos(n\pi/2)$ [1]

(d) $a_n = \frac{2^n}{n!}$ [1]

2

4 marks

Find a formula for a_n , starting at $n = 1$.

(a) $2, 4, 6, 8, \dots$ [1]

(b) $1, -1, 1, -1, \dots$ [1]

(c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ [1]

(d) $1, 4, 9, 16, \dots$ [1]

3

6 marks

Decide whether each sequence is strictly increasing, strictly decreasing, or non-monotonic.

(a) $a_n = 3 - \frac{1}{n}$ [1]

(b) $a_n = (-1)^n$ [1]

(c) $a_n = \frac{n}{n+1}$. Use the difference $a_{n+1} - a_n$. [2]

(d) $a_n = n^2 - 6n$ [2]

4

6 marks

Decide whether each sequence is bounded above, bounded below, both, or neither. Where a bound exists, give one.

(a) $a_n = \frac{1}{n}$ [1]

(b) $a_n = (-1)^n n$ [2]

(c) $a_n = \frac{2n+1}{n+1}$ [2]

(d) $a_n = n^2$ [1]

5

6 marks

(a) Let $a_n = \frac{3-n}{n}$. Find m and M with $m \leq a_n \leq M$ for all n , and then a K with $|a_n| \leq K$, using $K = \max\{|m|, |M|\}$. [3]

(b) Let $b_n = \sqrt{n+4} - \sqrt{n}$. Multiplying by $\sqrt{n+4} + \sqrt{n}$, show that $0 \leq b_n \leq \frac{2}{\sqrt{n}}$, and deduce that (b_n) is bounded. [3]

6

6 marks

(a) Let $a_n = 3n - 2$. For $C = 100$, find the smallest $N \in \mathbb{N}$ such that $a_n > C$ for all $n > N$. [3]

(b) Let $a_n = n^2$. For $C = 1000$, find the smallest such N . [2]

(c) For $a_n = n^2$ and a general $C > 0$, give an N that works. [1]

7

6 marks

Decide whether each sequence tends to ∞ , tends to $-\infty$, or neither.

(a) $a_n = n - 10$ [1]

(b) $a_n = -n^2$ [1]

(c) $a_n = (-1)^n n$ [1]

(d) $a_n = n \sin(n\pi/2)$ [1]

(e) $a_n = 5 - \sqrt{n}$ [1]

(f) $a_n = n + (-1)^n$ [1]

8

6 marks

Use the comparison rule (if $b_n \geq a_n$ and $a_n \rightarrow \infty$ then $b_n \rightarrow \infty$) or the sum, product and multiple rules for sequences tending to infinity. Name the rule you use.

- (a) $a_n = n^2 + n$ [1]
- (b) $a_n = n^3 - n^2$. Compare it with $n - 1$. [2]
- (c) $a_n = -4n$ [1]
- (d) $a_n = (n + 1)(2n - 3)$ [2]

9

6 marks

For each sequence, decide whether it tends to infinity, and justify your answer, using a subsequence or a comparison.

- (a) $a_n = n \cos(n\pi)$ [2]
- (b) $b_n = 2^n \sin(n\pi/2)$ [2]
- (c) $c_n = n + 2 \sin n$ [2]

10

7 marks

Let $a_n = \frac{n^2}{n+1}$.

- (a) Show that $a_n \geq \frac{n}{2}$ for all $n \in \mathbb{N}$. [2]
- (b) Use (a) to explain why $(a_n) \rightarrow \infty$, and use it to find an N that works for $C = 50$. [3]
- (c) Find the smallest N that works for $C = 50$. [2]

11

8 marks

Exam style. Let $a_n = \frac{n^2 + 1}{n}$.

- (a) Write down a_1, a_2, a_3, a_4 . [1]
- (b) Show that (a_n) is strictly increasing by simplifying $a_{n+1} - a_n$. [3]
- (c) Is (a_n) bounded below? Bounded above? Give a bound where one exists. [2]
- (d) For $C = 1000$, find the smallest N such that $a_n > C$ for all $n > N$. [2]

12

8 marks

Exam style. Let $a_n = n^2 - 10n + 30$.

- (a) Write down $a_1, \dots, a_7$. [1]
- (b) Is (a_n) monotonic? Find all n with $a_{n+1} > a_n$, and say from where it is increasing. [3]
- (c) By completing the square, find the largest lower bound of (a_n) . Is it bounded above? [2]
- (d) For $C = 100$, find the smallest N such that $a_n > C$ for all $n > N$. [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.