

Topic 3 · Sheet A: Worked answers and mark scheme

Sequences: monotonic, bounded and tending to infinity

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topics 1 to 2

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 First terms

4 marks

(a) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}$. (b) $-2, -2, 0, 4, 10$. (c) $0, -1, 0, 1, 0$. (d) $2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}$.

B1 each part correct

4

Common mistake Starting at $n = 0$. The notes start every sequence at $n = 1$.

2 Formulas for sequences

4 marks

(a) $2n$. (b) $(-1)^{n+1}$. (c) $\frac{1}{2^n}$. (d) n^2 .

B1 each part correct

4

Why this question In (b), $(-1)^n$ would start with -1 . The exponent $n + 1$ fixes the first term.

3 Monotonic or not

6 marks

- (a) **Strictly increasing**: $\frac{1}{n}$ decreases, so $3 - \frac{1}{n}$ increases.
- (b) **Non-monotonic**: $-1, 1, -1, \dots$
- (c) $a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0$: **strictly increasing**.
- (d) $-5, -8, -9, -8, -5, \dots$: it falls then rises, so **non-monotonic**.

B1	(a)	1
B1	(b)	1
M1	(c) the difference as one fraction	1
A1	(c)	1
M1	(d) listing terms or the difference $2n - 5$	1
A1	(d)	1

Why this question Part (d) is not monotonic, but it is increasing from $n = 3$ on. "Eventually" properties like this come up in Topic 5.

4 Bounds

6 marks

- (a) **Both**: $0 < \frac{1}{n} \leq 1$.
- (b) **Neither**: the even terms $2, 4, 6, \dots$ are unbounded above and the odd terms $-1, -3, -5, \dots$ unbounded below.
- (c) $\frac{2n+1}{n+1} = 2 - \frac{1}{n+1}$, so **both**: $\frac{3}{2} \leq a_n < 2$.
- (d) **Bounded below** by 1, **not above**.

B1	(a)	1
B1	(b) neither	1
E1	(b) reason for both directions	1
M1	(c) writing $2 - \frac{1}{n+1}$	1
A1	(c)	1
B1	(d)	1

Why this question Bounds are not unique: any number at least 2 is an upper bound in (c). The rewriting in (c) makes the best bounds easy to see.

5 Finding bounds

6 marks

- (a) $a_n = \frac{3}{n} - 1$, which decreases from $a_1 = 2$ and stays above -1 . So $m = -1$, $M = 2$, and $K = \max\{|m|, |M|\} = 2$.
- (b) $b_n = \frac{(\sqrt{n+4} - \sqrt{n})(\sqrt{n+4} + \sqrt{n})}{\sqrt{n+4} + \sqrt{n}} = \frac{4}{\sqrt{n+4} + \sqrt{n}} \leq \frac{4}{2\sqrt{n}} = \frac{2}{\sqrt{n}}$. Also $b_n \geq 0$. So $0 \leq b_n \leq 2$, and (b_n) is bounded.

B1	(a) m	1
B1	(a) M	1
B1	(a) K	1
M1	(b) multiplying by $\sqrt{n+4} + \sqrt{n}$	1
M1	(b) using $\sqrt{n+4} \geq \sqrt{n}$ in the denominator	1
A1	(b) $0 \leq b_n \leq 2$	1

Why this question The “multiply by the conjugate” trick in (b) is a standard move, and it comes back in Topic 4 to show such sequences are null.

6 Finding N for a given C

6 marks

- (a) $3n - 2 > 100$ means $n > 34$. So $N = 34$: for $n > 34$, $a_n \geq 103$. ($N = 33$ fails, since $a_{34} = 100$.)
- (b) $31^2 = 961$ and $32^2 = 1024$, so $N = 31$.
- (c) Any $N \geq \sqrt{C}$ works: if $n > N \geq \sqrt{C}$ then $n^2 > C$.

M1	(a) solving $3n - 2 > 100$	1
A1	(a) $N = 34$	1
B1	(a) checking that 33 fails	1
M1	(b)	1
A1	(b)	1
B1	(c)	1

Common mistake In (a), giving $N = 35$ or $N = 33$. The condition is $n > N$, so $N = 34$ means $n \geq 35$.

7 Infinity, minus infinity, or neither

6 marks

- (a) ∞ . (b) $-\infty$. (c) **Neither**: the signs alternate. (d) **Neither**: $a_{2n} = 0$ for all n . (e) $-\infty$. (f) ∞ : $a_n \geq n - 1$.

B1	each part correct	6
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Why this question In part (d), one subsequence stuck at 0 is enough to stop a sequence tending to infinity.

8 Using the rules

6 marks

- (a) ∞ : $n^2 \rightarrow \infty$ and $n \rightarrow \infty$, so the sum does.
- (b) $n^3 - n^2 = n^2(n - 1) \geq n - 1$ for $n \geq 1$, and $n - 1 \rightarrow \infty$. So ∞ by comparison.
- (c) $-\infty$: a negative multiple of a sequence tending to ∞ tends to $-\infty$.
- (d) Both $n + 1 \rightarrow \infty$ and $2n - 3 \rightarrow \infty$, so ∞ by the product rule.

B1	(a) with reason	1
M1	(b) the comparison $n^3 - n^2 \geq n - 1$	1
A1	(b)	1
B1	(c)	1
M1	(d) both factors tend to infinity	1
A1	(d) product rule	1

Common mistake In (b), arguing " $\infty - \infty = 0$ ". Infinity is not a number; a difference needs a comparison instead.

9 Justifying divergence

6 marks

- (a) $a_n = (-1)^n n$. The odd terms are negative, so for $C = 1$ no N works. **It does not tend to infinity.**
- (b) $b_{2n} = 2^{2n} \sin(n\pi) = 0$ for all n , so **it does not tend to infinity.**
- (c) $c_n \geq n - 2$, and $n - 2 \rightarrow \infty$, so $(c_n) \rightarrow \infty$ by comparison.

B1	(a) no	1
E1	(a) negative terms beyond any N	1
B1	(b) no	1
E1	(b) the subsequence $b_{2n} = 0$	1
M1	(c) $\sin n \geq -1$	1
A1	(c)	1

Why this question To show a sequence does *not* tend to infinity you need one C for which every N fails, which a bad subsequence provides.

10 A lower bound gives an N

7 marks

- (a) $n + 1 \leq 2n$ for $n \geq 1$, so $\frac{n^2}{n+1} \geq \frac{n^2}{2n} = \frac{n}{2}$.
- (b) $\frac{n}{2} \rightarrow \infty$, so $(a_n) \rightarrow \infty$ by comparison. For $C = 50$: $\frac{n}{2} > 50$ when $n > 100$, so $N = 100$ works.
- (c) $a_{50} = \frac{2500}{51} \approx 49.0$ and $a_{51} = \frac{2601}{52} \approx 50.02$, and (a_n) is increasing. So $N = 50$.

M1	(a) $n + 1 \leq 2n$	1
A1	(a)	1
B1	(b) comparison	1
M1	(b) solving $\frac{n}{2} > 50$	1
A1	(b) $N = 100$	1
M1	(c) testing values near 50	1
A1	(c) $N = 50$	1

Why this question The definition only asks for *some* N . A simple lower bound gives a valid N quickly, even if it is not the smallest one.

11 Exam style: an increasing sequence

8 marks

- (a) $2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}$.
- (b) $a_n = n + \frac{1}{n}$, so $a_{n+1} - a_n = 1 + \frac{1}{n+1} - \frac{1}{n} = 1 - \frac{1}{n(n+1)} \geq 1 - \frac{1}{2} > 0$. **Strictly increasing.**
- (c) **Bounded below** by $a_1 = 2$. **Not bounded above**, since $a_n > n$.
- (d) $n + \frac{1}{n} > 1000$ first holds at $n = 1000$, where $a_{1000} = 1000.001$; and $a_{999} < 1000$. So $N = 999$.

B1	(a)	1
M1	(b) writing $a_n = n + \frac{1}{n}$	1
M1	(b) simplifying the difference	1
A1	(b) showing it is positive	1
B1	(c) lower bound	1
B1	(c) not bounded above, with reason	1
M1	(d)	1
A1	(d) $N = 999$	1

Why this question Splitting the fraction in (b) is the move that makes everything else easy. It is worth trying whenever the top has higher degree than the bottom.

12 Exam style: eventually increasing

8 marks

- (a) 21, 14, 9, 6, 5, 6, 9.
- (b) **Not monotonic.** $a_{n+1} - a_n = 2n - 9 > 0$ exactly when $n \geq 5$, so it is increasing from a_5 on.
- (c) $a_n = (n - 5)^2 + 5 \geq 5$, with equality at $n = 5$. So 5 is the largest lower bound. **Not bounded above.**
- (d) $(n - 5)^2 + 5 > 100$ means $(n - 5)^2 > 95$, so $n - 5 \geq 10$. The smallest n is 15, so $N = 14$.

B1	(a)	1
B1	(b) not monotonic	1
M1	(b) $a_{n+1} - a_n = 2n - 9$	1
A1	(b) $n \geq 5$	1
M1	(c) completing the square	1
A1	(c)	1
M1	(d)	1
A1	(d)	1

Why this question Monotonicity and tending to infinity are different properties: this sequence tends to infinity although it starts by decreasing.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.