

Topic 2 · Sheet A: Questions

Absolute values and the triangle inequality

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topic 1

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the absolute value and its properties, the interval property and the triangle inequality with its equality case, plus Topic 1.

Notation. $|x|$ is the absolute value of x . By Remark 2.9, $|x - a|$ is the distance from x to a . Give every solution set as an interval or a union of intervals.

1

4 marks

Evaluate each expression.

- (a) $|-7|$ [1]
- (b) $|3 - 8|$ [1]
- (c) $||-2| - 5|$ [1]
- (d) $|(-3) \cdot 4|$ [1]

2

4 marks

Use the interval property, $|x - a| < b \iff a - b < x < a + b$, to write each set as an interval.

- (a) $\{x \in \mathbb{R} : |x| < 3\}$ [1]
- (b) $\{x \in \mathbb{R} : |x - 2| < 1\}$ [1]
- (c) $\{x \in \mathbb{R} : |x + 4| < 2\}$ [1]
- (d) $\{x \in \mathbb{R} : |2x - 1| < 5\}$ [1]

3

6 marks

Write each open interval in the form $\{x \in \mathbb{R} : |x - a| < b\}$, giving a and b .

- (a) $(1, 7)$ [2]
- (b) $(-5, 1)$ [2]
- (c) $(-0.1, 0.3)$ [2]

4

6 marks

Solve each inequality.

- (a) $|x - 3| > 2$ [2]
- (b) $|x + 1| \geq 4$ [2]
- (c) $|3 - x| \leq 1$ [2]

5

6 marks

- (a) With $x = -3$ and $y = 5$, compute $|xy|$ and $|x||y|$. [1]
- (b) With $x = -6$ and $y = 4$, compute $|x/y|$ and $|x|/|y|$. [1]
- (c) Compute $|| - 2.5||$. [1]
- (d) For each pair $(x, y) = (3, 4), (3, -4)$ and $(-2, -5)$, compare $|x + y|$ with $|x| + |y|$. In which cases does equality hold, and how does the equality case of the triangle inequality explain this? [3]

6

6 marks

Use the distance interpretation of $|x - a|$, or case analysis.

- (a) Solve $|x - 1| < |x + 3|$. [3]
- (b) Solve $|x| = |x - 4|$. [1]
- (c) Solve $|x - 2| \leq |x|$. [2]

7

6 marks

Solve each inequality by considering cases.

- (a) $|2x - 3| < x$ [3]
- (b) $|x^2 - 4| < 5$ [3]

8

6 marks

Use the triangle inequality in each part.

- (a) Suppose $|x - 2| < 1$. Find a number K with $|x + 5| < K$. [2]
- (b) Suppose $|x - 2| < 1$. Show that $|x^2 - 4| < 5$, by first bounding $|x + 2|$. [2]
- (c) Suppose $|a - 1| < 0.01$ and $|b - 3| < 0.02$. Find a bound for $|(a + b) - 4|$. [2]

9

6 marks

Use the equality case of the triangle inequality: $|x + y| = |x| + |y|$ exactly when $xy \geq 0$.

- (a) For which x is $|x + 2| = |x| + 2$? [2]
- (b) For which x is $|x - 1| = |x| + 1$? [2]
- (c) For which x is $|x| + |x - 3| = 3$? [2]

10

7 marks

Let $h(x) = |x - 1| + |x + 2|$.

- (a) Write $h(x)$ without absolute values on each of the intervals $(-\infty, -2)$, $[-2, 1]$ and $(1, \infty)$, and hence solve $h(x) < 5$. [4]
- (b) Explain why $h(x) < 3$ has no solutions. [2]
- (c) Solve $h(x) = 3$. [1]

11

8 marks

Exam style.

- (a) Solve $|x^2 - 1| < 3$. [3]
- (b) Solve $|x^2 - 1| > x + 1$. Treat $x < -1$, $x = -1$ and $x > -1$ separately. [4]
- (c) Give real numbers x and y with $|x + y| < |x| + |y|$. [1]

12

8 marks

Exam style. This question previews the ϵ - δ ideas of Topic 12.

- (a) Find a $\delta > 0$ such that $|x - 3| < \delta$ implies $|x^2 - 9| < 0.7$. Start by writing $|x^2 - 9| = |x - 3| |x + 3|$ and assuming $\delta \leq 1$. [4]
- (b) Check your δ with one value of x close to 3. [1]
- (c) Suppose $|x - 3| < 0.1$. Find the interval in which x^2 must lie. [3]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.