

Topic 2 · Sheet A: Worked answers and mark scheme

Absolute values and the triangle inequality

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Topic 1

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Evaluating absolute values

4 marks

(a) **7**. (b) **5**. (c) $|2 - 5| = \mathbf{3}$. (d) **12**.

B1 each part correct

4

Common mistake In (c), working from the outside in. Always evaluate the innermost absolute value first.

2 The interval property

4 marks

(a) $(-3, 3)$. (b) $(1, 3)$. (c) $|x - (-4)| < 2$: $(-6, -2)$. (d) $-5 < 2x - 1 < 5$ gives $(-2, 3)$.

B1 each part correct

4

Why this question In part (c), $|x + 4|$ is the distance from x to -4 .

3 Intervals as distances

6 marks

The centre is the midpoint and the radius is half the length.

- (a) $|x - 4| < 3$: $a = 4$, $b = 3$.
(b) $|x + 2| < 3$: $a = -2$, $b = 3$.
(c) $|x - 0.1| < 0.2$: $a = 0.1$, $b = 0.2$.

B1	each centre a	3
B1	each radius b	3

Why this question Going from an interval back to $|x - a| < b$ is a skill used constantly in the definition of convergence: $|a_n - a| < \epsilon$ means $a_n \in (a - \epsilon, a + \epsilon)$.

4 Greater-than inequalities

6 marks

- (a) The points more than 2 from 3: $(-\infty, 1) \cup (5, \infty)$.
(b) The points at least 4 from -1 : $(-\infty, -5] \cup [3, \infty)$.
(c) $|3 - x| = |x - 3|$, so the points within 1 of 3: $[2, 4]$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b) closed brackets	1
M1	(c) using $ 3 - x = x - 3 $	1
A1	(c)	1

Common mistake Writing $|x - 3| > 2$ as $-2 > x - 3 > 2$, which no x satisfies. A “greater than” inequality gives two separate pieces.

5 Checking the properties

6 marks

- (a) $|xy| = |-15| = 15$ and $|x||y| = 3 \cdot 5 = 15$. **Equal.**
(b) $|x/y| = |-1.5| = 1.5$ and $|x|/|y| = 6/4 = 1.5$. **Equal.**
(c) **2.5.**
(d) $(3, 4)$: $7 = 7$. $(3, -4)$: $1 < 7$. $(-2, -5)$: $7 = 7$. **Equality for $(3, 4)$ and $(-2, -5)$** , where x and y have the same sign, exactly as the equality case of the triangle inequality says.

B1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d) all three comparisons	1
B1	(d) equality in the two same-sign cases	1
E1	(d) same sign, citing the equality case	1

Why this question Checking a theorem on numbers before using it in general is a good habit. Part (d) shows the triangle inequality is sharp.

6 Comparing distances

6 marks

- (a) x is closer to 1 than to -3 . The midpoint of -3 and 1 is -1 , so $(-1, \infty)$. (By algebra: squaring both non-negative sides gives $-2x + 1 < 6x + 9$, so $x > -1$.)
- (b) Equal distance from 0 and 4: $x = 2$.
- (c) Closer to 2 than to 0, or equally close: $[1, \infty)$.

M1	(a) distance or squaring method	1
M1	(a) finding the boundary -1	1
A1	(a)	1
B1	(b)	1
M1	(c) boundary at 1	1
A1	(c)	1

Why this question Squaring is safe here because both sides are non-negative. This is the same observation used to prove the triangle inequality.

7 Absolute values with case analysis

6 marks

- (a) We need $x > 0$. Then $|2x - 3| < x$ means $-x < 2x - 3 < x$. The left part gives $x > 1$ and the right part gives $x < 3$: $(1, 3)$.
- (b) $-5 < x^2 - 4 < 5$ means $x^2 > -1$, always true, and $x^2 < 9$: $(-3, 3)$.

M1	(a) using the interval property with $b = x$, needing $x > 0$	1
M1	(a) solving both inequalities	1
A1	(a)	1
M1	(b) splitting into two inequalities	1
M1	(b) seeing $x^2 > -1$ always holds	1
A1	(b)	1

Common mistake In (a), using the interval property with $b = x$ without checking $x > 0$. For $x \leq 0$ there are no solutions, since $|2x - 3| \geq 0$.

8 Bounding with the triangle inequality

6 marks

- (a) $|x + 5| = |(x - 2) + 7| \leq |x - 2| + 7 < 8$. So $K = 8$ works.
- (b) $|x + 2| = |(x - 2) + 4| \leq |x - 2| + 4 < 5$, so $|x^2 - 4| = |x - 2| |x + 2| < 1 \cdot 5 = 5$.
- (c) $|(a + b) - 4| = |(a - 1) + (b - 3)| \leq |a - 1| + |b - 3| < 0.03$.

M1	(a) writing $x + 5 = (x - 2) + 7$	1
A1	(a)	1
M1	(b) bounding $ x + 2 < 5$	1
A1	(b)	1
M1	(c) regrouping as $(a - 1) + (b - 3)$	1
A1	(c)	1

Why this question Adding and subtracting the right number is the key trick in nearly every limit argument for sequences and continuity.

9 When the triangle inequality is an equality

6 marks

- (a) Equality when x and 2 have the same sign (or $x = 0$): $[0, \infty)$.
- (b) $|x + (-1)| = |x| + |-1|$ when x and -1 have the same sign: $(-\infty, 0]$.
- (c) $|x| + |3 - x| = 3 = |x + (3 - x)|$ when x and $3 - x$ have the same sign: $[0, 3]$.

M1	(a)	1
A1	(a)	1
M1	(b) writing $x - 1$ as $x + (-1)$	1
A1	(b)	1
M1	(c) writing $3 = x + (3 - x)$	1
A1	(c)	1

Common mistake Forgetting that 0 counts as having “the same sign” as anything here. The condition in the proof is $xy \geq 0$.

10 A sum of absolute values

7 marks

- (a) On $(-\infty, -2)$: $h(x) = -2x - 1$. On $[-2, 1]$: $h(x) = 3$. On $(1, \infty)$: $h(x) = 2x + 1$. Then $-2x - 1 < 5$ gives $x > -3$; $3 < 5$ always; $2x + 1 < 5$ gives $x < 2$. So $(-3, 2)$.

- (b) By the triangle inequality, $h(x) = |1 - x| + |x + 2| \geq |(1 - x) + (x + 2)| = 3$. **So $h(x) \geq 3$ always.**
- (c) From (a), $[-2, 1]$.

M1	(a) splitting into three cases	1
A1	(a) the three formulas	1
M1	(a) solving on each piece	1
A1	(a)	1
M1	(b) triangle inequality, or the formulas from (a)	1
A1	(b) $h(x) \geq 3$	1
B1	(c)	1

Why this question Geometrically, $h(x)$ is the total distance from x to 1 and to -2 . That is at least 3, the distance between them, with equality between the two points.

11 Exam style: quadratics inside absolute values

8 marks

- (a) $-3 < x^2 - 1 < 3$ means $x^2 > -2$, always true, and $x^2 < 4$: $(-2, 2)$.
- (b) For $x < -1$: $x + 1 < 0 \leq |x^2 - 1|$, so it holds. At $x = -1$: $0 > 0$ is false. For $x > -1$: $|x^2 - 1| = |x - 1|(x + 1) > x + 1$ becomes $|x - 1| > 1$, so $x < 0$ or $x > 2$. Together: $(-\infty, -1) \cup (-1, 0) \cup (2, \infty)$.
- (c) For example $x = 2, y = -1$: $|x + y| = 1 < 3$.

M1	(a) splitting into two inequalities	1
M1	(a) seeing $x^2 > -2$ always holds	1
A1	(a)	1
B1	(b) the case $x < -1$	1
B1	(b) the case $x = -1$ excluded	1
M1	(b) dividing by $x + 1 > 0$ to get $ x - 1 > 1$	1
A1	(b)	1
B1	(c)	1

Common mistake In (b), dividing by $x + 1$ without first checking its sign. This only works once you know $x + 1 > 0$.

12 Exam style: choosing a δ

8 marks

- (a) If $\delta \leq 1$ and $|x - 3| < \delta$, then $|x + 3| = |(x - 3) + 6| < 1 + 6 = 7$. So $|x^2 - 9| < 7\delta$. We need $7\delta \leq 0.7$: $\delta = 0.1$ works (any $\delta \leq 0.1$ does).
- (b) For example $x = 3.09$: $x^2 - 9 = 0.5481 < 0.7$. **It checks.**

(c) $x \in (2.9, 3.1)$, and since $x > 0$, squaring keeps the order: $x^2 \in (8.41, 9.61)$.

M1	(a) factorising $ x^2 - 9 $	1
M1	(a) bounding $ x + 3 < 7$ using $\delta \leq 1$	1
M1	(a) requiring $7\delta \leq 0.7$	1
A1	(a) $\delta = 0.1$ or smaller	1
B1	(b)	1
M1	(c) $x \in (2.9, 3.1)$	1
M1	(c) squaring is order-preserving for positive numbers	1
A1	(c)	1

Why this question This is exactly the calculation behind showing $x \mapsto x^2$ is ϵ - δ continuous at 3. Seeing it with numbers first makes the ϵ - δ definition much less abstract.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.