

Topic 1 · Sheet A: Worked answers and mark scheme

Inequalities: the order on $\mathbb{R}$, graphs and case analysis

Suggested time

About 75 minutes

Total marks

73

Questions

12

Builds on

Nothing earlier

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Three kinds of inequality

4 marks

(a) **Always true:** $x^2 \geq 0$. (b) **Never true:** subtracting x gives $1 < 0$. (c) **Sometimes:** true for $x = 3$, false for $x = 0$. (d) **Never true:** a sum of squares is at least 0.

B1 each part correct

4

Why this question Inequalities split into these three types. Spotting types (2) and (3) quickly saves effort later.

2 Interval notation

4 marks

(a) $(-2, 5]$. (b) $[3, \infty)$. (c) $(-\infty, 1) \cup (4, \infty)$. (d) $(\frac{1}{2}, 1)$.

B1 each part, with the correct brackets

4

Common mistake Using a square bracket at ∞ . Infinity is never included, so it always gets a round bracket.

3 Linear inequalities

6 marks

- (a) Add 5 (compatibility with addition), then multiply by $\frac{1}{3} > 0$ (compatibility with positive multiplication): $x < 4$, so $(-\infty, 4)$.
- (b) Add -2 (compatibility with addition): $-3x \geq 6$. Multiplying by the negative number $-\frac{1}{3}$ reverses the inequality: $x \leq -2$, so $(-\infty, -2]$.
- (c) Add -2 throughout: $-6 < 2x \leq 8$. Multiply by $\frac{1}{2}$: $(-3, 4]$.
- (d) Add $-x$: $-1 \leq 5$, which always holds. **All of $\mathbb{R}$.**

B1	(a)	1
M1	(b) reversing the inequality when multiplying by a negative	1
A1	(b)	1
M1	(c) treating both inequalities at once	1
A1	(c) correct brackets	1
B1	(d)	1

Common mistake In (b), forgetting to flip the sign when multiplying by a negative number.

4 Reading solutions off graphs

6 marks

- (a) $y = x^2$ and $y = 2x$ meet where $x^2 = 2x$, at $x = 0$ **and** $x = 2$. The parabola lies below the line between them: $(0, 2)$.
- (b) $y = x^3$ and $y = x$ meet at $x = -1, 0, 1$. The cubic is on or above the line on $[-1, 0] \cup [1, \infty)$.
- (c) $y = \frac{1}{x}$ and $y = 1$ meet at $x = 1$. For $x < 0$, $\frac{1}{x} < 0 < 1$. So $(0, 1)$.

B1	(a) meeting points	1
A1	(a)	1
B1	(b) meeting points	1
A1	(b) closed brackets included	1
B1	(c) meeting point	1
A1	(c), excluding negative x	1

Why this question The graph method works well when the meeting points are easy to find, and it is a good check on the algebra of case analysis.

5 Case analysis for products

6 marks

- (a) Both factors positive gives $x > 2$; both negative gives $x < -3$. So $(-\infty, -3) \cup (2, \infty)$.
- (b) The factors have opposite signs or one is zero: $[-3, 2]$.

(c) Factorise: $(x - 1)(x - 4) < 0$, so $(1, 4)$.

M1	(a) the two sign cases	1
A1	(a)	1
M1	(b) adapting the cases for $\leq$	1
A1	(b)	1
M1	(c) factorising first	1
A1	(c)	1

Why this question Part (b) is part (a) with the inequality reversed and the endpoints added. Together they cover all of $\mathbb{R}$, which is a quick check.

6 Sign tables

6 marks

- (a) Zeros at $-2, 1, 4$. The signs on the four intervals, from the left, are $-, +, -, +$. So $(-2, 1) \cup (4, \infty)$.
- (b) $(x - 3)^2 \geq 0$ always, and equals 0 only at $x = 3$. So the product is ≥ 0 exactly when $x \geq 0$ or $x = 3$: $[0, \infty)$.

M1	(a) zeros in order	1
M1	(a) a correct sign table	1
A1	(a)	1
M1	(b) noticing $(x - 3)^2$ never changes sign	1
M1	(b) reducing to $x \geq 0$, keeping $x = 3$	1
A1	(b)	1

Common mistake In (b), treating $(x - 3)^2$ like two separate sign changes at 3. A repeated factor does not change sign.

7 Quotients

6 marks

- (a) Numerator and denominator have opposite signs: $(-1, 2)$.
- (b) Same signs, or numerator zero, and $x \neq -1$: $(-\infty, -1) \cup [3, \infty)$.
- (c) Same signs: $(-\infty, 0) \cup (1, \infty)$.

M1	(a)	1
A1	(a)	1
M1	(b)	1
A1	(b) -1 excluded, 3 included	1
M1	(c)	1
A1	(c)	1

Why this question A quotient has the same sign as the product of its factors, except that the denominator can never be zero.

8 True or false?

6 marks

- (a) **False:** $a = -3, b = 1$ gives $a^2 = 9 > 1 = b^2$.
- (b) **True:** multiplying $a < b$ by $a > 0$ gives $a^2 < ab$, and by $b > 0$ gives $ab < b^2$ (compatibility with positive multiplication). Then $a^2 < b^2$ by transitivity.
- (c) **True:** adding $-a - b$ gives $-b < -a$ (compatibility with addition), and multiplying by $2 > 0$ gives $-2b < -2a$ (compatibility with positive multiplication).

B1	(a) false	1
B1	(a) valid counterexample	1
B1	(b) true	1
E1	(b) the properties named	1
B1	(c) true	1
E1	(c) the properties named	1

Why this question Squaring is only safe for positive numbers. That is why the proof of the triangle inequality in Topic 2 is careful to square only non-negative quantities.

9 One fraction, then a sign table

7 marks

- (a) $\frac{1}{x+1} - \frac{1}{x+4} = \frac{3}{(x+1)(x+4)} < 0$, so $(x+1)(x+4) < 0$: $(-4, -1)$.
- (b) $\frac{2}{x-1} - \frac{1}{x+1} = \frac{2(x+1) - (x-1)}{(x-1)(x+1)} = \frac{x+3}{(x-1)(x+1)} \geq 0$. With points $-3, -1, 1$ the signs are $-, +, -, +$. Including the zero at $x = -3$: $[-3, -1) \cup (1, \infty)$.

M1	(a) single fraction	1
M1	(a) reducing to the sign of the denominator	1
A1	(a)	1
M1	(b) single fraction $\frac{x+3}{(x-1)(x+1)}$	1
M1	(b) sign table with $-3, -1, 1$	1
A1	(b) correct intervals	1
A1	(b) -3 included, ± 1 excluded	1

Common mistake Multiplying both sides by $(x+1)(x+4)$. Its sign depends on x , which is exactly the trap to avoid with rational inequalities.

10 A full sign table

6 marks

- (a) $g(0) = \frac{(-1)(-2)}{(3)(-4)} = -\frac{1}{6} < 0$. **Yes.**
- (b) The points are $-3, 1, 2, 4$. From the left the signs are $+, -, +, -, +$, so $(-3, 1) \cup (2, 4)$.
- (c) For example $x + 3 \leq x - 2$.

B1	(a)	1
M1	(b) the four points in order	1
M1	(b) a sign table with four factors	1
A2	(b) both intervals. <i>A1 for one</i>	2
B1	(c)	1

Why this question Part (a) finds one solution, $x = 0$, and the sign table finds all of them at once.

11 Exam style: polynomial and rational inequalities

8 marks

- (a) $x^3 - x^2 - 6x = x(x - 3)(x + 2)$. Points $-2, 0, 3$, signs $-, +, -, +$: $(-2, 0) \cup (3, \infty)$.
- (b) $\frac{(x - 1)(x + 1)}{(x - 2)(x + 2)}$. Points $-2, -1, 1, 2$, signs $+, -, +, -, +$. Include the zeros ± 1 , exclude ± 2 : $(-2, -1] \cup [1, 2)$.
- (c) For example $x = -\frac{5}{2}$: then $x + 3 > 0$ but $x + 2 < 0$.

M1	(a) factorising	1
M1	(a) sign table	1
A1	(a)	1
M1	(b) factorising top and bottom	1
M1	(b) sign table with four points	1
A1	(b) correct intervals	1
A1	(b) endpoints correct	1
B1	(c)	1

Why this question Part (c) is an important warning: a one-way step can lose solutions, so it matters which of your steps are equivalences.

12 Exam style: combining inequalities

8 marks

- (a) $x^2 < x$ means $x(x - 1) < 0$, so $x \in (0, 1)$. $x < x^3$ means $x(x - 1)(x + 1) > 0$, so $x \in (-1, 0) \cup (1, \infty)$. These sets do not meet: **no x works.**

(b) $\frac{1}{x} - x = \frac{1 - x^2}{x} < 0$, which is $\frac{(x - 1)(x + 1)}{x} > 0$: $(-1, 0) \cup (1, \infty)$.

(c) $(-1, 0) \cup (1, 2)$.

M1	(a) solving $x^2 < x$	1
A1	(a) $(0, 1)$	1
A1	(a) $(-1, 0) \cup (1, \infty)$	1
A1	(a) intersection empty	1
M1	(b) single fraction	1
M1	(b) sign table	1
A1	(b)	1
B1	(c)	1

Why this question Part (a) shows an inequality can have no solutions even when it is not obviously of type (3). The empty answer comes out of careful case work.

Total: 73 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.