

Topic 15 · Sheet A: Questions

Homomorphisms, isomorphisms, cosets and quotients

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 14

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need homomorphisms, isomorphisms, roots of unity, images, kernels, cosets, quotients and the theorems of Jordan and Lagrange, plus Topics 1 to 14.

Notation. $\varphi : G \rightarrow H$ is a homomorphism when $\varphi(g \cdot g') = \varphi(g) \cdot \varphi(g')$ for all $g, g' \in G$, and an isomorphism when it is also a bijection; then $G \cong H$. $\text{imag}(\varphi) = \varphi(G)$ and $\text{kern}(\varphi) = \{g \in G \mid \varphi(g) = e_H\}$. $\zeta_n^k = \exp(2\pi i k/n)$ and $U_n = \{\zeta_n^k \mid k \in \mathbb{Z}\}$. A left coset is $gH = \{gh \mid h \in H\}$ and a right coset is Hg ; in an additive group we write $g + H$. G/H is the set of left cosets. $C_n = (\mathbb{Z}/n\mathbb{Z}, +)$, Rot_n is the rotation subgroup of D_{2n} , and products in $\text{Sym}(n)$ are compositions, right to left, as on Topic 14.

1

4 marks

Decide whether each function is a homomorphism. If not, give a pair g, g' where it fails.

- (a) $\varphi : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}, +)$, $\varphi(k) = 3k$ [1]
- (b) $\varphi : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}, +)$, $\varphi(k) = k + 1$ [1]
- (c) $\varphi : (\mathbb{Z}, +) \rightarrow C_5$, $\varphi(k) = [k]_5$ [1]
- (d) $\varphi : (\mathbb{R}, +) \rightarrow (\mathbb{R}, +)$, $\varphi(x) = x^2$ [1]

2

4 marks

Let $\psi : C_{12} \rightarrow C_6$ be $\psi([k]_{12}) = [k]_6$.

- (a) Find $\psi([7]_{12})$ and $\psi([11]_{12})$. [1]
- (b) Check that $\psi([5]_{12} + [9]_{12}) = \psi([5]_{12}) + \psi([9]_{12})$. [1]
- (c) Find $\text{kern}(\psi)$. [1]
- (d) Find $\text{imag}(\psi)$. [1]

3

6 marks

Let $\varphi : C_4 \rightarrow C_8$ be $\varphi([k]_4) = [2k]_8$.

- (a) Find $\varphi([k]_4)$ for each $k \in \mathbb{Z}_4$. [1]
- (b) Check that $\varphi([3]_4 + [2]_4) = \varphi([3]_4) + \varphi([2]_4)$. [1]
- (c) Find $\text{imag}(\varphi)$ and $\text{kern}(\varphi)$. [2]
- (d) Is φ injective? Is it an isomorphism? [2]

4

6 marks

Let $\varphi : (\mathbb{Z}, +) \rightarrow D_8$ be the homomorphism pow_g , with $g = R_{\pi/2}$, so $\varphi(n) = R_{\pi/2}^n$. Use $\varphi(g^k) = \varphi(g)^k$.

- (a) Find $\varphi(0)$ and $\varphi(3)$. [2]
- (b) Find $\varphi(-1)$ and $\varphi(10)$. [2]
- (c) Find $\text{imag}(\varphi)$. [1]
- (d) Find $\text{kern}(\varphi)$. [1]

5

6 marks

- (a) List the elements of U_4 in the form $a + bi$. [1]
- (b) Use $\zeta_{pn}^{pk} = \zeta_n^k$ to evaluate ζ_8^2 , ζ_{12}^3 and ζ_6^3 . [2]
- (c) Write out the isomorphism $\varphi([k]_4) = \zeta_4^k$, as a table of values. [1]
- (d) Check that $\varphi([3]_4 + [3]_4) = \varphi([3]_4) \cdot \varphi([3]_4)$. [1]
- (e) Find the order of $-i$ in U_4 , and of $[3]_4$ in C_4 . [1]

6

6 marks

Decide whether each pair of groups is isomorphic. If it is, give an isomorphism; if not, give a property that one has and the other does not.

- (a) C_6 and $\text{Sym}(3)$ [2]
- (b) C_4 and Rot_4 [1]
- (c) C_4 and $G = \{[1]_5, [2]_5, [3]_5, [4]_5\}$ under multiplication modulo 5 [2]
- (d) Rot_4 and the subgroup $K = \{R_0, R_\pi, M_0, M_\pi\}$ of D_8 [1]

7

6 marks

Find the image and kernel of each homomorphism.

- (a) $p_6 : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}, +)$, $p_6(k) = 6k$ [2]
- (b) $q_6 : (\mathbb{Z}, +) \rightarrow C_6$, $q_6(k) = [k]_6$ [2]
- (c) $\text{pari} : \text{Sym}(3) \rightarrow \mathbb{Z}/2\mathbb{Z}$ [2]

8

6 marks

Let $G = C_{12}$ and $H = \langle 4 \rangle$.

- (a) List H , and list every coset $k + H$, without repeats. [3]
- (b) Is $5 + H = 1 + H$? Explain using the fact that cosets form a partition. [1]
- (c) Find $|G/H|$, and check that $|G| = |H| \cdot |G/H|$. [2]

9

6 marks

Let $G = \text{Sym}(3)$ and $H = \{\epsilon, (0\ 1)\}$. Use your multiplication table from Topic 14.

- (a) Find the left coset gH for every $g \in G$, and list G/H . [3]
- (b) Find the right coset $H(0\ 2)$. Is it equal to $(0\ 2)H$? [2]
- (c) How many left cosets are there? Check this against Lagrange's theorem. [1]

10

7 marks

Use Lagrange's theorem: the order of a subgroup, and of an element, divides the order of the group.

- (a) A group G has order 12. What orders could a subgroup of G have? [1]
- (b) Can D_8 contain an element of order 3? [1]
- (c) List the subgroups of C_7 , and the order of each non-zero element. Explain how Lagrange's theorem decides this. [3]
- (d) Can $\text{Sym}(4)$ have a subgroup of order 5? [1]
- (e) Find the order of $(0\ 1\ 2)(3\ 4)$ in $\text{Sym}(5)$, and check that it divides $|\text{Sym}(5)|$. [1]

11

9 marks

Exam style. Let $\psi : C_6 \rightarrow C_3$ be $\psi([k]_6) = [k]_3$.

- (a) $[1]_6 = [7]_6$. Check that ψ gives the same answer for both representatives. [1]
- (b) Write out $\psi([k]_6)$ for each $k \in \llbracket 6 \rrbracket$. [1]
- (c) Check that $\psi([4]_6 + [5]_6) = \psi([4]_6) + \psi([5]_6)$. [1]
- (d) Find $\text{kern}(\psi)$ and $\text{imag}(\psi)$. [2]
- (e) List the cosets of $K = \text{kern}(\psi)$ in C_6 , and check that there are $|C_6|/|K|$ of them. [3]
- (f) What do the elements of a single coset of K have in common? [1]

12

10 marks

Exam style. This question builds an explicit isomorphism between D_6 and $\text{Sym}(3)$. Place the vertices v_0, v_1, v_2 of an equilateral triangle at angles $0, 2\pi/3$ and $4\pi/3$ on the unit circle. Recall that R_θ sends the point at angle α to angle $\alpha + \theta$, and M_θ sends it to angle $\theta - \alpha$. Each $g \in D_6$ permutes the vertices; let $\Phi(g) \in \text{Sym}(3)$ send k to j when $g(v_k) = v_j$.

- (a) Find $\Phi(R_0)$, $\Phi(R_{2\pi/3})$ and $\Phi(R_{4\pi/3})$ in cycle notation. [2]

- (b) Find $\Phi(M_0)$, $\Phi(M_{2\pi/3})$ and $\Phi(M_{4\pi/3})$ in cycle notation, showing where each vertex goes. [4]
- (c) Check that $\Phi(R_{2\pi/3} \circ M_0) = \Phi(R_{2\pi/3}) \Phi(M_0)$. [3]
- (d) Which elements of D_6 are sent to even permutations? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.