

Topic 15 · Sheet A: Worked answers and mark scheme

Homomorphisms, isomorphisms, cosets and quotients

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 14

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Homomorphism or not?

4 marks

(a) **Yes:** $3(k + \ell) = 3k + 3\ell$. (b) **No:** $\varphi(0 + 0) = 1$ but $\varphi(0) + \varphi(0) = 2$.

(c) **Yes:** $[k + \ell]_5 = [k]_5 + [\ell]_5$. (d) **No:** $\varphi(1 + 1) = 4$ but $\varphi(1) + \varphi(1) = 2$.

B1 each part, with a failing pair where needed

4

Common mistake A quick test: a homomorphism must send the identity to the identity. Part (b) fails this at once, since $\varphi(0) = 1$.

2 Reducing from C_{12} to C_6

4 marks

(a) $[1]_6$ **and** $[5]_6$. (b) $[5]_{12} + [9]_{12} = [2]_{12} \mapsto [2]_6$, and $[5]_6 + [3]_6 = [8]_6 = [2]_6$. **They agree.**

(c) $\{[0]_{12}, [6]_{12}\}$. (d) **All of C_6 .**

B1 each part correct

4

Why this question Part (b) is one instance of the homomorphism property. A single check proves nothing in general, but it is how you catch an error in an example.

3 Doubling from C_4 into C_8

6 marks

- (a) $[0]_8, [2]_8, [4]_8, [6]_8$ for $k = 0, 1, 2, 3$.
- (b) $[3]_4 + [2]_4 = [1]_4 \mapsto [2]_8$, and $[6]_8 + [4]_8 = [10]_8 = [2]_8$. **They agree.**
- (c) $\text{imag}(\varphi) = \{0, 2, 4, 6\}$ and $\text{kern}(\varphi) = \{[0]_4\}$.
- (d) **Injective: yes**, as the four values are different. **Not an isomorphism:** it is not surjective, since $[1]_8$ is missed.

B1	(a)	1
B1	(b)	1
B1	(c) image	1
B1	(c) kernel	1
B1	(d) injective	1
E1	(d) not an isomorphism, with reason	1

Why this question The image is a subgroup of C_8 , as images always are: it is $\langle 2 \rangle$. The kernel is trivial exactly because φ is injective.

4 Powers of a rotation

6 marks

- (a) $\varphi(0) = R_0$ and $\varphi(3) = R_{3\pi/2}$.
- (b) $\varphi(-1) = R_{\pi/2}^{-1} = R_{3\pi/2}$ and $\varphi(10) = R_{5\pi} = R_{\pi}$.
- (c) $\text{imag}(\varphi) = \{R_0, R_{\pi/2}, R_{\pi}, R_{3\pi/2}\} = \text{Rot}_4$.
- (d) $\text{kern}(\varphi) = 4\mathbb{Z}$, the multiples of 4, since $R_{\pi/2}$ has order 4.

B1	(a) $\varphi(0)$	1
B1	(a) $\varphi(3)$	1
B1	(b) $\varphi(-1)$	1
B1	(b) $\varphi(10)$	1
B1	(c)	1
B1	(d)	1

Why this question The image of pow_g is $\langle g \rangle$. The kernel tells you the order of g : it is $k\mathbb{Z}$ where k is the order.

5 Roots of unity

6 marks

- (a) $1, i, -1, -i$.
- (b) $\zeta_8^2 = \zeta_4^1 = i; \zeta_{12}^3 = \zeta_4^1 = i; \zeta_6^3 = \zeta_2^1 = -1$.
- (c) $[0]_4 \mapsto 1, [1]_4 \mapsto i, [2]_4 \mapsto -1, [3]_4 \mapsto -i$.
- (d) $[3]_4 + [3]_4 = [2]_4 \mapsto -1$, and $(-i)(-i) = i^2 = -1$. **They agree.**
- (e) **Both have order 4**, as an isomorphism preserves orders.

B1	(a)	1
B2	(b) all three. <i>B1 for two</i>	2
B1	(c)	1
B1	(d)	1
B1	(e)	1

Why this question U_4 is a group of complex numbers under multiplication, and C_4 is a group of classes under addition, yet they are the same group. That is what isomorphism means.

6 Isomorphic or not?

6 marks

- (a) **No**: C_6 is commutative but $\text{Sym}(3)$ is not. (Also, C_6 has an element of order 6 and $\text{Sym}(3)$ does not.)
- (b) **Yes**: $[k]_4 \mapsto R_{2\pi k/4}$.
- (c) **Yes**: $[k]_4 \mapsto [2^k]_5$, giving $0 \mapsto 1, 1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 3$. It is a bijection, and a homomorphism by the laws of exponents, as $2^4 \equiv 1$.
- (d) **No**: $R_{\pi/2} \in \text{Rot}_4$ has order 4, but every element of K has order 1 or 2.

B1	(a) answer	1
E1	(a) a distinguishing property	1
B1	(b)	1
M1	(c) using that 2 generates G	1
A1	(c) the explicit map	1
E1	(d)	1

Why this question To prove two groups are *not* isomorphic, find a group property that one has and the other lacks. Counting elements of each order is often the quickest.

7 Images and kernels

6 marks

- (a) $\text{imag}(p_6) = 6\mathbb{Z}$, $\text{kern}(p_6) = \{0\}$. (b) $\text{imag}(q_6) = C_6$, $\text{kern}(q_6) = 6\mathbb{Z}$.
 (c) $\text{imag}(\text{pari}) = \mathbb{Z}/2\mathbb{Z}$ (a transposition is odd), $\text{kern}(\text{pari}) = \{\epsilon, (0\ 1\ 2), (0\ 2\ 1)\}$, the even permutations.

B1	each image	3
B1	each kernel	3

Why this question Parts (a) and (b) are the standard examples, and the kernel in (c) is the subgroup you found in question 11 of Topic 14. Kernels are always subgroups.

8 Cosets in C_{12}

6 marks

- (a) $H = \{0, 4, 8\}$. The cosets are $\{0, 4, 8\}, \{1, 5, 9\}, \{2, 6, 10\}, \{3, 7, 11\}$.
 (b) **Yes**: $5 \in 1 + H$ and $5 \in 5 + H$. The cosets form a partition, so two cosets that meet are equal.
 (c) $|G/H| = 4$, and $|H| \cdot |G/H| = 3 \cdot 4 = 12 = |G|$.

B1	(a) H	1
B2	(a) all four cosets. <i>B1 for two or three</i>	2
E1	(b)	1
B1	(c) $ G/H $	1
B1	(c) check	1

Why this question The cosets of $\langle 4 \rangle$ group the elements of C_{12} by their remainder mod 4. So $C_{12}/\langle 4 \rangle$ behaves like C_4 .

9 Cosets in $\text{Sym}(3)$

6 marks

- (a) $\epsilon H = (0\ 1)H = \{\epsilon, (0\ 1)\}$, $(0\ 2)H = (0\ 1\ 2)H = \{(0\ 2), (0\ 1\ 2)\}$, $(1\ 2)H = (0\ 2\ 1)H = \{(1\ 2), (0\ 2\ 1)\}$. So G/H has these three cosets.
 (b) $H(0\ 2) = \{(0\ 2), (0\ 1)(0\ 2)\} = \{(0\ 2), (0\ 2\ 1)\}$. **Not equal** to $(0\ 2)H$.
 (c) **3**, and $|H| \cdot 3 = 6 = |\text{Sym}(3)|$.

M1	(a) computing gh for $h \in H$	1
A2	(a) all three distinct cosets. <i>A1 for two</i>	2
B1	(b) right coset	1
B1	(b) not equal	1
B1	(c)	1

Common mistake Multiplying on the wrong side. In gH the g is on the left, so $(0\ 2)H$ contains $(0\ 2)(0\ 1) = (0\ 1\ 2)$, while $H(0\ 2)$ contains $(0\ 1)(0\ 2) = (0\ 2\ 1)$.

10 Jordan and Lagrange

7 marks

- (a) The divisors of 12: $1, 2, 3, 4, 6, 12$.
- (b) **No**: 3 does not divide $|D_8| = 8$.
- (c) $\{0\}$ and C_7 . **Every non-zero element has order 7**, since 7 is prime.
- (d) **No**: 5 does not divide $|\text{Sym}(4)| = 24$.
- (e) **6**, and 6 divides $|\text{Sym}(5)| = 120$.

B1	(a)	1
B1	(b)	1
B1	(c) subgroups	1
B1	(c) orders	1
E1	(c) 7 is prime, so subgroup orders can only be 1 or 7	1
B1	(d)	1
B1	(e)	1

Common mistake Reading Lagrange's theorem backwards. It says what orders are *possible*; it does not promise a subgroup of every order dividing $|G|$.

11 Exam style: a homomorphism and its kernel

9 marks

- (a) $\psi([1]_6) = [1]_3$ and $\psi([7]_6) = [7]_3 = [1]_3$. **The same**, as 3 divides 6.
- (b) $[0]_3, [1]_3, [2]_3, [0]_3, [1]_3, [2]_3$ for $k = 0, \dots, 5$.
- (c) $[4]_6 + [5]_6 = [3]_6 \mapsto [0]_3$, and $[1]_3 + [2]_3 = [0]_3$. **They agree**.
- (d) $\text{kern}(\psi) = \{[0]_6, [3]_6\}$ and $\text{imag}(\psi) = C_3$.
- (e) $\{0, 3\}, \{1, 4\}, \{2, 5\}$.
- (f) **They have the same image under ψ** , so there is one coset for each element of the image.

B1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d) kernel	1
B1	(d) image	1
M1	(e) adding each element to K	1
A1	(e)	1
B1	(e) $3 = 6/2$	1
E1	(f)	1

Why this question Part (f) is the key idea behind quotients: $C_6/\text{kern}(\psi)$ matches up with the image C_3 . Part (a) is the well-definedness check.

12 Exam style: D_6 and $\text{Sym}(3)$

10 marks

- (a) $\Phi(R_0) = \epsilon$, $\Phi(R_{2\pi/3}) = (0\ 1\ 2)$, $\Phi(R_{4\pi/3}) = (0\ 2\ 1)$.
- (b) M_0 : angles $0, \frac{2\pi}{3}, \frac{4\pi}{3} \mapsto 0, -\frac{2\pi}{3}, -\frac{4\pi}{3}$, which are v_0, v_2, v_1 . So $\Phi(M_0) = (1\ 2)$. Similarly $\Phi(M_{2\pi/3}) = (0\ 1)$ and $\Phi(M_{4\pi/3}) = (0\ 2)$.
- (c) By the composition rules, $R_{2\pi/3} \circ M_0 = M_{2\pi/3}$, so the left side is $(0\ 1)$. The right side is $(0\ 1\ 2)(1\ 2) = (0\ 1)$, from the $\text{Sym}(3)$ table. **They agree**.
- (d) **The rotations** $R_0, R_{2\pi/3}, R_{4\pi/3}$.

B2	(a) all three. B1 for two	2
M1	(b) using $\alpha \mapsto \theta - \alpha$	1
A3	(b) all three, with the vertex images shown. A2 for two; A1 for one	3
B1	(c) left side, via the composition rules	1
M1	(c) composing the permutations right to left	1
A1	(c) right side $(0\ 1)$	1
B1	(d)	1

Why this question Φ is a bijection and a homomorphism, so $D_6 \cong \text{Sym}(3)$, and its inverse is an explicit isomorphism $\text{Sym}(3) \rightarrow D_6$. Part (d) matches Rot_3 with the kernel of pari : rotations keep the orientation of the triangle, and reflections reverse it.

Total: 76 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.