

Topic 14 · Sheet A: Questions

Permutations: the symmetric group, cycle notation and parity

Suggested time

About 75 minutes

Total marks

77

Questions

12

Builds on

Topics 1 to 13

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need $\text{Sym}(n)$, two-line, one-line and cycle notation, composing permutations, orbit decompositions, the cycle-finding algorithm, inversions and parity, plus Topics 1 to 13.

Notation. One-line notation is written in square brackets, so $\sigma = [1 \ 2 \ 0]$ means $\sigma(0) = 1$, $\sigma(1) = 2$, $\sigma(2) = 0$. Products are compositions: $\sigma\tau = \sigma \circ \tau$, so $(\sigma\tau)(i) = \sigma(\tau(i))$ and τ acts *first*. A product of cycles such as $(0 \ 1 \ 2)(2 \ 3)$ is composed the same way, right to left. The identity is ϵ . $O_\sigma(k)$ is the forward orbit of k , and O_σ is the orbit decomposition. $D(n)$ is the set of pairs $\{a, b\} \subset \llbracket n \rrbracket$ with $a \neq b$; σ has an inversion at $\{a, b\}$ with $a < b$ when $\sigma(b) < \sigma(a)$. $\text{inv}(\sigma)$ counts inversions and $\text{pari}(\sigma) = [\text{inv}(\sigma)]_2$. You may use $\text{pari}(\tau\sigma) = \text{pari}(\tau) + \text{pari}(\sigma)$, and that a k -cycle has parity $[k - 1]_2$.

1

4 marks

Let $\sigma \in \text{Sym}(5)$ have two-line notation $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 0 & 2 \end{pmatrix}$.

- (a) Write σ in one-line notation. [1]
- (b) What is $\sigma(\sigma(0))$? [1]
- (c) List the fixed points of σ . [1]
- (d) Is $[1 \ 2 \ 2 \ 0 \ 3]$ the one-line notation of a permutation in $\text{Sym}(5)$? [1]

2

4 marks

In $\text{Sym}(4)$, let $\sigma = [1 \ 2 \ 3 \ 0]$ and $\tau = [1 \ 0 \ 2 \ 3]$.

- (a) Find $\sigma\tau$ in one-line notation. [1]
- (b) Find $\tau\sigma$ in one-line notation. [1]
- (c) Do σ and τ commute? [1]
- (d) Find σ^2 . [1]

3

6 marks

In $\text{Sym}(6)$, let $\sigma = [2 \ 0 \ 1 \ 5 \ 3 \ 4]$ and $\tau = [0 \ 2 \ 1 \ 4 \ 3 \ 5]$.

- (a) Find $\sigma\tau$ in one-line notation. [2]
- (b) Find $\tau\sigma$ in one-line notation. [2]
- (c) Find σ^{-1} in one-line notation, and check that $\sigma\sigma^{-1} = \epsilon$. [2]

4

6 marks

Let $\sigma = [1 \ 2 \ 0 \ 4 \ 3]$ in $\text{Sym}(5)$.

- (a) Write out the one-line notation of every power $\sigma, \sigma^2, \sigma^3, \dots$ until you reach ϵ . [4]
- (b) What is the order of σ ? Compare this with the order of $[1 \ 2 \ 0 \ 4 \ 5 \ 3]$ in $\text{Sym}(6)$. [2]

5

6 marks

Let $\sigma = [3 \ 0 \ 4 \ 1 \ 2 \ 5]$ and $\tau = [0 \ 3 \ 2 \ 4 \ 1 \ 5]$ in $\text{Sym}(6)$.

- (a) Find $O_\sigma(0)$ and $O_\sigma(2)$. [2]
- (b) Write down O_σ , and check that it is a partition of $\llbracket 6 \rrbracket$. [1]
- (c) Is σ a k -cycle for some k ? [1]
- (d) Show that τ is a k -cycle. Give k , its support and its cycle notation. [2]

6

6 marks

Write each permutation in cycle notation, first with its fixed points and then with them omitted.

- (a) $[4 \ 0 \ 1 \ 2 \ 3 \ 5]$ in $\text{Sym}(6)$ [2]
- (b) $[2 \ 5 \ 0 \ 4 \ 3 \ 1 \ 6]$ in $\text{Sym}(7)$ [2]
- (c) $[1 \ 3 \ 5 \ 0 \ 2 \ 4 \ 7 \ 6]$ in $\text{Sym}(8)$ [2]

7

6 marks

- (a) Write $(0 \ 3)(1 \ 4 \ 2)$ in $\text{Sym}(5)$ in one-line notation. [1]
- (b) Write $(0 \ 1 \ 2)(2 \ 3)$ in $\text{Sym}(4)$ as a single cycle. [2]
- (c) Write $(0 \ 1)(0 \ 2)$ in $\text{Sym}(3)$ as a single cycle. [1]
- (d) Find the inverse of $(0 \ 1 \ 2 \ 3)$ in cycle notation. [1]
- (e) What is the order of $(0 \ 1 \ 2)(3 \ 4)$? [1]

8

6 marks

Run the cycle-finding algorithm on $\sigma = [2 \ 4 \ 0 \ 3 \ 1]$ in $\text{Sym}(5)$.

- (a) Make a table with one row per symbol appended to C , recording the symbol, which step of the algorithm appended it, and the set W afterwards. [4]
- (b) State the final string C , and give the cycle notation with fixed points omitted. [2]

9

6 marks

Let $\sigma = [2 \ 0 \ 3 \ 1]$ in $\text{Sym}(4)$.

- (a) List the six pairs in $D(4)$. [1]
- (b) Find every pair at which σ has an inversion. [2]
- (c) Find $\text{inv}(\sigma)$ and $\text{pari}(\sigma)$. [1]
- (d) Find σ^{-1} , and check that it has the same number of inversions as σ . [2]

10

7 marks

- (a) Check that $(0 \ 1 \ 2 \ 3) = (0 \ 3)(0 \ 2)(0 \ 1)$ in $\text{Sym}(4)$, by finding the one-line notation of each side. [2]
- (b) Find inv of $(0 \ 1 \ 2 \ 3)$ by counting inversions, and check that its parity agrees with the number of transpositions in part (a). [2]
- (c) Use the parity of a k -cycle and the sum rule for parities to find the parity of $(0 \ 1)(2 \ 3 \ 4)$ and of $(0 \ 1 \ 2)(3 \ 4 \ 5 \ 6 \ 7)$. [3]

11

9 marks

Exam style. The six elements of $\text{Sym}(3)$ are ϵ , $(0 \ 1)$, $(0 \ 2)$, $(1 \ 2)$, $(0 \ 1 \ 2)$ and $(0 \ 2 \ 1)$.

- (a) Give the full multiplication table of $\text{Sym}(3)$ in cycle notation, with the elements in the order above. The entry in row σ , column τ is $\sigma\tau$. [4]
- (b) Is $\text{Sym}(3)$ commutative? Give a reason. [1]
- (c) Find the order of each element. [2]
- (d) Find the parity of each element. Do the even permutations form a subgroup? [2]

12

11 marks

Exam style. Let $\sigma = [3 \ 5 \ 2 \ 4 \ 0 \ 1]$ in $\text{Sym}(6)$.

- (a) Write σ in cycle notation, with and without fixed points. [2]
- (b) Write out σ^k in cycle notation for $k = 1, \dots, 6$, and give the order of σ . [3]
- (c) Find σ^{-1} in one-line and in cycle notation. [2]
- (d) Use the parity of a k -cycle and the sum rule for parities to find $\text{pari}(\sigma)$. [1]
- (e) Count the inversions of σ , listing the pairs, and check that your answer agrees with (d). [3]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.