

Topic 14 · Sheet A: Worked answers and mark scheme

Permutations: the symmetric group, cycle notation and parity

Suggested time

About 75 minutes

Total marks

77

Questions

12

Builds on

Topics 1 to 13

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Two-line and one-line notation

4 marks

(a) $\begin{bmatrix} 3 & 1 & 4 & 0 & 2 \end{bmatrix}$. (b) $\sigma(0) = 3$ and $\sigma(3) = 0$, so 0. (c) 1 only. (d) **No**: 2 appears twice and 4 never appears, so it is not a bijection.

B1 each part correct

4

Why this question A list is the one-line notation of a permutation exactly when it contains each element of $\llbracket n \rrbracket$ once. That is injectivity and surjectivity from Topic 2.

2 Composing in $\text{Sym}(4)$

4 marks

(a) $\sigma\tau(i) = \sigma(\tau(i))$: $\sigma(1), \sigma(0), \sigma(2), \sigma(3)$, giving $\begin{bmatrix} 2 & 1 & 3 & 0 \end{bmatrix}$. (b) $\tau(1), \tau(2), \tau(3), \tau(0)$, giving $\begin{bmatrix} 0 & 2 & 3 & 1 \end{bmatrix}$.
(c) **No**: (a) and (b) differ. (d) $\begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix}$.

B1 each part correct

4

Common mistake Applying σ first in $\sigma\tau$. The notes compose like functions: τ acts first. Here doing it the wrong way round swaps the answers to (a) and (b).

3 Composing in $\text{Sym}(6)$

6 marks

- (a) $\sigma\tau = [2 \ 1 \ 0 \ 3 \ 5 \ 4]$.
- (b) $\tau\sigma = [1 \ 0 \ 2 \ 5 \ 4 \ 3]$.
- (c) Swap the two rows of the two-line notation and re-sort: $\sigma(1) = 0, \sigma(2) = 1, \sigma(0) = 2, \sigma(4) = 3, \sigma(5) = 4, \sigma(3) = 5$. So $\sigma^{-1} = [1 \ 2 \ 0 \ 4 \ 5 \ 3]$. Then $\sigma(\sigma^{-1}(i)) = i$ for each i .

M1	(a) evaluating $\sigma(\tau(i))$	1
A1	(a)	1
M1	(b) evaluating $\tau(\sigma(i))$	1
A1	(b)	1
B1	(c) σ^{-1}	1
B1	(c) the check	1

Why this question To invert from one-line notation, ask “which i is sent to 0?”, then “which i is sent to 1?”, and so on.

4 Powers and order

6 marks

- (a) $\sigma = [1 \ 2 \ 0 \ 4 \ 3], \sigma^2 = [2 \ 0 \ 1 \ 3 \ 4], \sigma^3 = [0 \ 1 \ 2 \ 4 \ 3], \sigma^4 = [1 \ 2 \ 0 \ 3 \ 4], \sigma^5 = [2 \ 0 \ 1 \ 4 \ 3], \sigma^6 = \epsilon$.
- (b) The order is **6**. The permutation $[1 \ 2 \ 0 \ 4 \ 5 \ 3]$ in $\text{Sym}(6)$ has order 3: it moves 0, 1, 2 round in a loop of length 3 and 3, 4, 5 in another. Here the loops have lengths 3 and 2, and both must return home, so we need a multiple of 6.

M1	(a) composing with σ each time	1
A3	(a) all five non-trivial powers. <i>A2 for four; A1 for three</i>	3
B1	(b) order	1
E1	(b) comparison	1

Why this question This previews the next questions: the order depends on the lengths of the cycles, which cycle notation makes visible.

5 Orbit decompositions

6 marks

- (a) $0 \mapsto 3 \mapsto 1 \mapsto 0$, so $O_\sigma(0) = \{0, 1, 3\}$; and $2 \mapsto 4 \mapsto 2$, so $O_\sigma(2) = \{2, 4\}$.
- (b) $O_\sigma = \{\{0, 1, 3\}, \{2, 4\}, \{5\}\}$. The parts are non-empty, disjoint and cover $\llbracket 6 \rrbracket$.
- (c) **No**: two parts have more than one element.

- (d) $O_\tau = \{\{0\}, \{1, 3, 4\}, \{2\}, \{5\}\}$. Exactly one part is not a singleton, so τ is a **3-cycle** with support $\{1, 3, 4\}$ and cycle notation $(1\ 3\ 4)$.

B1	(a) $O_\sigma(0)$	1
B1	(a) $O_\sigma(2)$	1
B1	(b)	1
B1	(c)	1
M1	(d) finding O_τ	1
A1	(d) k , support and notation	1

Why this question O_σ is always a partition, just like equivalence classes in Topic 7, and you can read off “ k -cycle” straight from it.

6 Cycle notation

6 marks

- (a) $(0\ 4\ 3\ 2\ 1)(5)$, or $(0\ 4\ 3\ 2\ 1)$.
 (b) $(0\ 2)(1\ 5)(3\ 4)(6)$, or $(0\ 2)(1\ 5)(3\ 4)$.
 (c) $(0\ 1\ 3)(2\ 5\ 4)(6\ 7)$, in both forms, since there are no fixed points.

B2	each part. <i>B1 if one cycle is wrong or the form with fixed points is missing</i>	6
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Common mistake Starting a cycle somewhere other than its smallest element, as in $(4\ 3\ 2\ 1\ 0)$, or listing the cycles in the wrong order. Each cycle starts at its smallest element, and cycles are listed by their smallest elements.

7 Cycles to one-line, and products of cycles

6 marks

- (a) $0 \mapsto 3, 3 \mapsto 0, 1 \mapsto 4, 4 \mapsto 2, 2 \mapsto 1$: $[3\ 4\ 1\ 0\ 2]$.
 (b) Apply $(2\ 3)$ first. $0 \mapsto 0 \mapsto 1, 1 \mapsto 1 \mapsto 2, 2 \mapsto 3 \mapsto 3, 3 \mapsto 2 \mapsto 0$. So $(0\ 1\ 2\ 3)$.
 (c) $0 \mapsto 2 \mapsto 2, 2 \mapsto 0 \mapsto 1, 1 \mapsto 1 \mapsto 0$: $(0\ 2\ 1)$.
 (d) Run the cycle backwards: $(0\ 3\ 2\ 1)$.
 (e) **6**.

B1	(a)	1
M1	(b) tracking each element through $(2\ 3)$ then $(0\ 1\ 2)$	1
A1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e)	1

Why this question Part (b) shows two cycles that overlap in one element merging into one longer cycle. Part (e) is the same situation as question 4, in cycle notation.

8 Running the cycle-finding algorithm

6 marks

(a)

row	symbol	step	W
1	(	(1)	$\emptyset$
2	0	(2)	$\{0\}$
3	2	$(3b) : \sigma(0) = 2 \notin W$	$\{0, 2\}$
4	)	$(3a) : \sigma(2) = 0 \in W$	$\{0, 2\}$
5	(	(1)	$\{0, 2\}$
6	1	(2)	$\{0, 1, 2\}$
7	4	$(3b) : \sigma(1) = 4 \notin W$	$\{0, 1, 2, 4\}$
8	)	$(3a) : \sigma(4) = 1 \in W$	$\{0, 1, 2, 4\}$
9	(	(1)	$\{0, 1, 2, 4\}$
10	3	(2)	$\{0, 1, 2, 3, 4\}$
11	)	$(3a) : \sigma(3) = 3 \in W$	$\{0, 1, 2, 3, 4\}$

(b) $C = (0\ 2)(1\ 4)(3)$; without fixed points, $(0\ 2)(1\ 4)$.

M1	(a) a new cycle starts at the smallest element not in W	1
M1	(a) closing a cycle when $\sigma(b) \in W$	1
A2	(a) all eleven rows. <i>A1 for one error</i>	2
B1	(b) C	1
B1	(b) without fixed points	1

Why this question Running the algorithm by hand once makes its two key tests clear: where each cycle starts, and when it closes.

9 Inversions

6 marks

- (a) $\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}$.
 (b) $\sigma(0), \dots, \sigma(3) = 2, 0, 3, 1$. Inversions at $\{0,1\}, \{0,3\}, \{2,3\}$.
 (c) $\text{inv}(\sigma) = 3$, so $\text{pari}(\sigma) = [1]_2$: σ is odd.
 (d) $\sigma^{-1} = [1 \ 3 \ 0 \ 2]$, with inversions at $\{0,2\}, \{1,2\}, \{1,3\}$: **also 3**.

B1	(a)	1
M1	(b) comparing $\sigma(a)$ and $\sigma(b)$ for each pair	1
A1	(b)	1
B1	(c)	1
B1	(d) σ^{-1}	1
A1	(d) three inversions	1

Why this question This checks on an example that σ and σ^{-1} have the same number of inversions. The inversions of σ^{-1} sit at different pairs, but there are just as many of them.

10 Parity of cycles

7 marks

- (a) Both sides are $[1 \ 2 \ 3 \ 0]$. For instance on the right, $0 \mapsto 1 \mapsto 1 \mapsto 1$ and $3 \mapsto 3 \mapsto 3 \mapsto 0$.
 (b) Inversions at $\{0,3\}, \{1,3\}, \{2,3\}$, so $\text{inv} = 3$, **odd**. Part (a) uses three transpositions, also odd, as the parity theorem predicts.
 (c) $(0 \ 1)(2 \ 3 \ 4)$: $[1]_2 + [2]_2 = [1]_2$, **odd**. $(0 \ 1 \ 2)(3 \ 4 \ 5 \ 6 \ 7)$: $[2]_2 + [4]_2 = [0]_2$, **even**.

M1	(a) composing right to left	1
A1	(a)	1
B1	(b) inv	1
E1	(b) agreement explained	1
M1	(c) adding the parities of the cycles	1
A1	(c) first	1
A1	(c) second	1

Common mistake Thinking a k -cycle has parity $[k]_2$. A k -cycle is a product of $k - 1$ transpositions, so a 5-cycle is even and a 4-cycle is odd.

11 Exam style: the multiplication table of $\text{Sym}(3)$

9 marks

(a)

	ϵ	(01)	(02)	(12)	(012)	(021)
ϵ	ϵ	(01)	(02)	(12)	(012)	(021)
(01)	(01)	ϵ	(021)	(012)	(12)	(02)
(02)	(02)	(012)	ϵ	(021)	(01)	(12)
(12)	(12)	(021)	(012)	ϵ	(02)	(01)
(012)	(012)	(02)	(12)	(01)	(021)	ϵ
(021)	(021)	(12)	(01)	(02)	ϵ	(012)

(b) **No:** $(0\ 1)(0\ 2) = (0\ 2\ 1)$ but $(0\ 2)(0\ 1) = (0\ 1\ 2)$.

(c) ϵ : **1**; each transposition: **2**; each 3-cycle: **3**.

(d) ϵ , $(0\ 1\ 2)$, $(0\ 2\ 1)$ **are even; the transpositions are odd.** The even ones **do form a subgroup**: where the rows and columns for ϵ , $(0\ 1\ 2)$ and $(0\ 2\ 1)$ meet, every entry is one of those three, and $(0\ 1\ 2)$ and $(0\ 2\ 1)$ are inverses of each other.

B4	(a) all 36 entries. <i>B3 for at most 2 errors; B2 for at most 4; B1 for at most 6</i>	4
B1	(b) with a non-commuting pair	1
B2	(c) all orders. <i>B1 for four or five</i>	2
B1	(d) parities	1
B1	(d) subgroup, with a reason	1

Why this question This is the full table of $\text{Sym}(3)$. It passes the sudoku check, and the even permutations are a copy of C_3 , much like Rot_3 inside D_6 on Topic 13.

12 Exam style: one permutation, every tool

11 marks

- (a) $(0\ 3\ 4)(1\ 5)(2)$, or $(0\ 3\ 4)(1\ 5)$.
- (b) $\sigma = (0\ 3\ 4)(1\ 5)$, $\sigma^2 = (0\ 4\ 3)$, $\sigma^3 = (1\ 5)$, $\sigma^4 = (0\ 3\ 4)$, $\sigma^5 = (0\ 4\ 3)(1\ 5)$, $\sigma^6 = \epsilon$. The order is **6**.
- (c) $\sigma^{-1} = [4\ 5\ 2\ 0\ 3\ 1] = (0\ 4\ 3)(1\ 5)$.
- (d) $\text{pari}(\sigma) = [3 - 1]_2 + [2 - 1]_2 = [1]_2$, **odd**.
- (e) Inversions at $\{0,2\}$, $\{0,4\}$, $\{0,5\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$: $\text{inv}(\sigma) = 11$, odd, which agrees.

B2	(a) both forms. <i>B1 for one</i>	2
M1	(b) powers of each cycle separately	1
A1	(b) all six powers	1
A1	(b) order 6	1
B1	(c) one-line	1
B1	(c) cycle notation	1
B1	(d)	1
M1	(e) comparing $\sigma(a)$ and $\sigma(b)$ for each of the 15 pairs	1
A1	(e) the eleven pairs, so $\text{inv}(\sigma) = 11$	1
B1	(e) odd, agreeing with (d)	1

Why this question Parts (d) and (e) reach the same answer in two very different ways. The cycle route takes a line; counting 11 inversions out of 15 pairs takes much longer, which is why the rule $\text{pari}(\tau\sigma) = \text{pari}(\tau) + \text{pari}(\sigma)$ is so useful.

Total: 77 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.