

Topic 13 · Sheet A: Questions

Groups: definitions, dihedral groups, subgroups and generators

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 12

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need the definition of a group, cyclic and dihedral groups, multiplication tables, commutativity, subgroups, generators, powers and orders, plus Topics 1 to 12.

Notation. $C_n = (\mathbb{Z}/n\mathbb{Z}, +)$; in it we write k for $[k]_n$. In the dihedral group D_{2n} , R_θ is rotation through θ and M_θ is reflection in the line at angle $\theta/2$, with θ a multiple of $2\pi/n$; angles are taken modulo 2π . They compose by these rules: $R_\eta \circ R_\theta = R_{\eta+\theta}$, $M_\eta \circ M_\theta = R_{\eta-\theta}$, $R_\eta \circ M_\theta = M_{\eta+\theta}$ and $M_\eta \circ R_\theta = M_{\eta-\theta}$. $\langle S \rangle$ is the subgroup generated by S , and the order of g is $|\langle g \rangle|$.

1

4 marks

Decide whether each is a group. If not, name a property that fails.

- (a) $(\mathbb{R}, +)$ [1]
- (b) $(\mathbb{Z}, \cdot)$ [1]
- (c) $(\mathbb{Q} - \{0\}, \cdot)$ [1]
- (d) $(\{0, 1\}, +)$, with ordinary addition [1]

2

4 marks

In C_6 :

- (a) What is the identity element? [1]
- (b) What is the negative of 2? [1]
- (c) Compute $5 + 4$. [1]
- (d) What is the order of the group? [1]

3

6 marks

Let $G = \{[1]_5, [2]_5, [3]_5, [4]_5\}$ with multiplication modulo 5.

- (a) Write out the multiplication table of G . [3]
- (b) Find the identity element, and the inverse of each element. [2]

- (c) Check that each row of your table is a permutation of G (the “sudoku” property). [1]

4

6 marks

Work in D_8 , where the angles are multiples of $\pi/2$. Write each answer as some R_θ or M_θ with $0 \leq \theta < 2\pi$.

- (a) $R_{\pi/2} \circ R_\pi$ [1]
- (b) $M_{\pi/2} \circ M_0$ [1]
- (c) $R_{\pi/2} \circ M_0$ [1]
- (d) $M_0 \circ R_{\pi/2}$ [1]
- (e) Do $R_{\pi/2}$ and M_0 commute? [1]
- (f) What is the order of the group D_8 ? [1]

5

6 marks

Use $R_\theta(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$ and $M_\theta(x, y) = (x \cos \theta + y \sin \theta, x \sin \theta - y \cos \theta)$.

- (a) Find $R_{\pi/2}(1, 0)$. [1]
- (b) Find $M_0(1, 2)$. [1]
- (c) Find $M_\pi(1, 2)$. [1]
- (d) Find $(R_{\pi/2} \circ M_0)(1, 2)$ and $M_{\pi/2}(1, 2)$, and check they agree with the composition rules. [2]
- (e) What is $M_\theta \circ M_\theta$, for any θ ? [1]

6

6 marks

- (a) List all the subgroups of C_8 . [3]
- (b) Is $\{0, 2, 5\}$ a subgroup of C_8 ? [1]
- (c) Is $\{0, 3, 6\}$ a subgroup of C_9 ? [1]
- (d) Find $\langle 8 \rangle$ in C_{12} , and its order. [1]

7

6 marks

Work in C_{12} .

- (a) Find the order of each element $k \in \llbracket 12 \rrbracket$. [3]
- (b) Which elements generate the whole of C_{12} ? [2]
- (c) Guess a formula for the order of k in terms of $\gcd(k, 12)$. [1]

8

6 marks

- (a) In $(\mathbb{R}_{>0}, \cdot)$, with $g = 2$, find g^5 and g^{-3} . [1]
- (b) In $(\mathbb{Z}, +)$, find the multiples $5 \cdot 3$ and $(-2) \cdot 3$. [1]
- (c) In D_8 , find $R_{\pi/2}^3$ and $R_{\pi/2}^{-1}$. [2]
- (d) In D_8 , find M_0^5 . [1]
- (e) What is the order of $R_{\pi/2}$ in D_8 ? [1]

9

6 marks

Work in C_{10} . Build $\langle S \rangle$ in layers: $H_0 = \{0\}$ and $H_{n+1} = H_n \cup (H_n + T)$, where $T = S \cup (-S)$.

- (a) With $S = \{4\}$, find $T = S \cup S^{-1}$, then H_0, H_1, H_2, H_3 , and hence $\langle 4 \rangle$. [3]
- (b) Find $\langle \{4, 5\} \rangle$. [2]
- (c) What is the order of 4? [1]

10

7 marks

Work in D_6 , where the angles are multiples of $2\pi/3$.

- (a) Compute $R_{2\pi/3} \circ M_0$ and $M_0 \circ R_{2\pi/3}$. Is D_6 commutative? [3]
- (b) Is the rotation subgroup $\text{Rot}_3 = \{R_0, R_{2\pi/3}, R_{4\pi/3}\}$ commutative? [1]
- (c) Write out the multiplication table of Rot_3 , and check that it has the sudoku property. [3]

11

10 marks

Exam style. Work in $D_6 = \{R_0, R_{2\pi/3}, R_{4\pi/3}, M_0, M_{2\pi/3}, M_{4\pi/3}\}$.

- (a) Write out the rows of the multiplication table for $R_{2\pi/3}, M_0$ and $M_{2\pi/3}$, with columns in the order listed above. (The entry in row g , column h is $g \circ h$.) [4]
- (b) Find the order of each element. [2]
- (c) List every subgroup of order 2, and explain why there are no others. [3]
- (d) Is $\{R_0, M_0, M_{2\pi/3}\}$ a subgroup? [1]

12

9 marks

Exam style. Let $G = \{[1]_7, \dots, [6]_7\}$ with multiplication modulo 7, writing k for $[k]_7$.

- (a) List the powers $3^1, 3^2, \dots, 3^6$, showing each step. What is the order of 3? [3]
- (b) What does (a) tell you about $\langle 3 \rangle$? [1]
- (c) Find the order of 2, and list $\langle 2 \rangle$. [2]
- (d) Find the inverse of each element of G . [2]
- (e) Is G commutative? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.