

Topic 13 · Sheet A: Worked answers and mark scheme

Groups: definitions, dihedral groups, subgroups and generators

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 12

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Group or not?

4 marks

(a) **Yes**, with identity 0. (b) **No**: 2 has no inverse, since $2x = 1$ has no integer solution. (c) **Yes**, with identity 1 and inverse $1/x$. (d) **No**: $1 + 1 = 2$ is not in the set, so $+$ is not a function $G \times G \rightarrow G$.

B1 each part, naming the failure where relevant

4

Common mistake Forgetting that the operation must stay inside G . The definition of a group asks for a function $G \times G \rightarrow G$ before anything else.

2 The cyclic group C_6

4 marks

(a) 0. (b) **4**, since $2 + 4 = 6 \equiv 0$. (c) **3**. (d) **6**.

B1 each part correct

4

Why this question In additive notation the identity is written 0 and the inverse of g is written $-g$. Here $-[2]_6 = [-2]_6 = [4]_6$.

3 A multiplication table

6 marks

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(b) The identity is 1. Inverses: $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$. (c) Each row contains 1, 2, 3, 4 exactly once. **Yes.**

B3	(a) all 16 entries. <i>B2 for 14 or 15; B1 for 12 or 13</i>	3
B1	(b) identity	1
B1	(b) all inverses	1
B1	(c)	1

Why this question To find an inverse from a table, look for the identity in that element's row. The "sudoku" property of group tables is a quick check on any table you write.

4 Composing in D_8

6 marks

(a) $R_{3\pi/2}$. (b) $R_{\pi/2}$. (c) $M_{\pi/2}$. (d) $M_{0-\pi/2} = M_{3\pi/2}$.
 (e) **No:** (c) and (d) differ. (f) **8:** four rotations and four reflections.

B1	each part correct	6
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Common mistake In (d), leaving the answer as $M_{-\pi/2}$ is fine in value, but the question asks for $0 \leq \theta < 2\pi$, so add 2π .

5 Rotations and reflections on points

6 marks

- (a) $(0, 1)$. (b) $(1, -2)$: M_0 is reflection in the x -axis. (c) $M_\pi(x, y) = (-x, y)$, so $(-1, 2)$.
 (d) $(R_{\pi/2} \circ M_0)(1, 2) = R_{\pi/2}(1, -2) = (2, 1)$, and $M_{\pi/2}(1, 2) = (0 \cdot 1 + 1 \cdot 2, 1 \cdot 1 - 0 \cdot 2) = (2, 1)$.
They agree, as $R_{\pi/2} \circ M_0 = M_{\pi/2}$.
 (e) By the composition rules, $M_\theta \circ M_\theta = R_{\theta-\theta} = R_0 = \text{Id}_{\mathbb{R}^2}$.

B1	(a)	1
B1	(b)	1
B1	(c)	1
M1	(d) both computations	1
A1	(d) both $(2, 1)$	1
B1	(e)	1

Why this question Checking one point is a quick way to catch a slip in the composition rules. Part (e) says that reflecting twice gets you back where you started.

6 Subgroups of cyclic groups

6 marks

- (a) $\{0\}$, $\{0, 4\}$, $\{0, 2, 4, 6\}$ and C_8 .
 (b) **No**: $2 + 2 = 4$ is not in it.
 (c) **Yes**: it contains 0, is closed under $+$, and 3 and 6 are negatives of each other.
 (d) $\{0, 4, 8\}$, of order 3.

B3	(a) all four. B2 for three; B1 for two	3
B1	(b)	1
B1	(c)	1
B1	(d)	1

Why this question Each subgroup of C_8 is generated by one element, $\langle k \rangle$, as always happens in a cyclic group.

7 Orders of elements in C_{12}

6 marks

- (a) For $k = 0, 1, \dots, 11$: 1, 12, 6, 4, 3, 12, 2, 12, 3, 4, 6, 12.
 (b) The elements of order 12: 1, 5, 7, 11.
 (c) **The order of k is $12/\gcd(k, 12)$.**

B3	(a) all twelve. <i>B2 for ten or eleven; B1 for eight or nine</i>	3
B2	(b) all four. <i>B1 for three</i>	2
B1	(c)	1

Why this question Part (b) links to Topic 12: k generates C_{12} exactly when k and 12 are relatively prime, which by Bézout means k has an inverse modulo 12.

8 Powers and multiples

6 marks

- (a) **32 and $1/8$.** (b) **15 and -6 .** (c) **$R_{3\pi/2}$ and $R_{3\pi/2}$.** (d) $M_0^2 = R_0$, so $M_0^5 = M_0$: M_0 . (e) **4.**

B1	(a)	1
B1	(b)	1
B1	(c) $R_{\pi/2}^3$	1
B1	(c) $R_{\pi/2}^{-1}$	1
B1	(d)	1
B1	(e)	1

Why this question In part (c) the cube and the inverse coincide because $R_{\pi/2}^4 = R_0$. In a group of finite order k , $g^{-1} = g^{k-1}$.

9 Building a generated subgroup

6 marks

- (a) $T = \{4, 6\}$, since $-4 = 6$. Then $H_0 = \{0\}$, $H_1 = \{0, 4, 6\}$, $H_2 = \{0, 2, 4, 6, 8\}$ and $H_3 = H_2$. So $\langle 4 \rangle = \{0, 2, 4, 6, 8\}$.
 (b) 5 is odd, and adding it to the evens gives the odds: **all of C_{10} .**
 (c) **5.**

B1	(a) T	1
M1	(a) $H_{n+1} = H_n \cup (H_n + T)$	1
A1	(a) the sets and $\langle 4 \rangle$	1
M1	(b) combining the two generators	1
A1	(b)	1
B1	(c)	1

Why this question The construction builds $\langle S \rangle$ in layers, words of length at most n . Once a layer repeats, nothing new can ever appear.

10 A non-commutative group

7 marks

(a) $R_{2\pi/3} \circ M_0 = M_{2\pi/3}$ and $M_0 \circ R_{2\pi/3} = M_{-2\pi/3} = M_{4\pi/3}$. **Not commutative.**

(b) **Yes:** $R_\eta \circ R_\theta = R_{\eta+\theta} = R_\theta \circ R_\eta$.

(c)

$\circ$	R_0	$R_{2\pi/3}$	$R_{4\pi/3}$
R_0	R_0	$R_{2\pi/3}$	$R_{4\pi/3}$
$R_{2\pi/3}$	$R_{2\pi/3}$	$R_{4\pi/3}$	R_0
$R_{4\pi/3}$	$R_{4\pi/3}$	R_0	$R_{2\pi/3}$

B1	(a) first composite	1
B1	(a) second composite	1
B1	(a) not commutative	1
B1	(b)	1
B2	(c) table. <i>B1 for one error</i>	2
B1	(c) each row and column contains each element once	1

Why this question Part (a) gives two elements of D_6 that do not commute. Every subgroup of a commutative group is commutative; part (b) shows the converse fails, since the non-commutative D_6 has the commutative subgroup Rot_3 .

11 Exam style: the dihedral group D_6

10 marks

Write R_k for $R_{2\pi k/3}$ and M_k for $M_{2\pi k/3}$. Columns $R_0, R_1, R_2, M_0, M_1, M_2$.

- (a) Row R_1 : $R_1, R_2, R_0, M_1, M_2, M_0$.
 Row M_0 : $M_0, M_2, M_1, R_0, R_2, R_1$.
 Row M_1 : $M_1, M_0, M_2, R_1, R_0, R_2$.
- (b) R_0 : **1**; R_1, R_2 : **3**; each M_k : **2**.
- (c) $\{R_0, M_0\}, \{R_0, M_{2\pi/3}\}, \{R_0, M_{4\pi/3}\}$.
- (d) **No**: $M_{2\pi/3} \circ M_0 = R_{2\pi/3}$ is not in it.

M1	(a) using the composition rules	1
A1	(a) row R_1	1
A1	(a) row M_0	1
A1	(a) row $M_{2\pi/3}$	1
B1	(b) rotations	1
B1	(b) reflections	1
B2	(c) all three. <i>B1 for one or two</i>	2
E1	(c) such a subgroup is $\{R_0, g\}$ with g of order 2, so g is a reflection	1
B1	(d)	1

Why this question Row M_0 swaps rotations and reflections: composing with a reflection always flips which kind of symmetry you have. This is why the reflections alone are never a subgroup.

12 Exam style: a cyclic group in disguise

9 marks

- (a) $3, 2, 6, 4, 5, 1$. The order of 3 is 6.
- (b) $\langle 3 \rangle = G$: so G is cyclic, generated by 3.
- (c) $2^1 = 2, 2^2 = 4, 2^3 = 8 \equiv 1$. Order **3**, and $\langle 2 \rangle = \{1, 2, 4\}$.
- (d) $1^{-1} = 1, 2^{-1} = 4, 3^{-1} = 5, 4^{-1} = 2, 5^{-1} = 3, 6^{-1} = 6$.
- (e) **Yes**: multiplication of integers is commutative, and so is multiplication modulo 7.

M1	(a) multiplying by 3 and reducing modulo 7 each time	1
A1	(a) powers	1
B1	(a) order 6	1
B1	(b)	1
B1	(c) order	1
B1	(c) subgroup	1
B2	(d) all six. <i>B1 for four or five</i>	2
B1	(e)	1

Why this question G looks nothing like C_6 , yet listing powers of 3 matches it with 0, 1, 2, 3, 4, 5 under addition. Spotting when two groups are “the same” is the idea of isomorphism, in Topic 15.

Total: 76 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.