

Topic 11 · Sheet A: Questions

Induction, recursion and the pigeonhole principle

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 10

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need recursion, partial sums, iterates and orbits, the well-ordering principle, prime factorisation, Fibonacci numbers and the pigeonhole principle plus Topics 1 to 10.

Notation. $f^{(n)}$ is the n th iterate of f , with $f^{(0)} = \text{Id}$ and $f^{(n+1)} = f^{(n)} \circ f$. The forward orbit of x is $O_f(x) = \{f^{(n)}(x) \mid n \in \mathbb{N}\}$, and x is a fixed point if $f(x) = x$. $\sum_{k=0}^{n-1} a_k$ is the partial sum $S(n)$. The Fibonacci numbers are $f(0) = 0$, $f(1) = 1$ and $f(k+2) = f(k+1) + f(k)$.

1

4 marks

- (a) Let $f(0) = 1$ and $f(n+1) = 3f(n) - 1$. Find $f(1), f(2), f(3), f(4)$. [2]
- (b) Let $g(0) = 0$ and $g(n+1) = g(n) + n$. Find $g(1), g(2), g(3), g(4)$. [2]

2

4 marks

- (a) Find $\sum_{k=0}^4 (2k+1)$. [1]
- (b) Find $\sum_{k=0}^5 k^2$. [1]
- (c) Let $a_k = 2^k$. Find $S(5) = \sum_{k=0}^4 a_k$, and guess a formula for $S(n)$. [2]

3

6 marks

- (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = 3x$. Find a formula for $f^{(n)}(x)$, and find $f^{(3)}(2)$. [2]

Let $g: \llbracket 6 \rrbracket \rightarrow \llbracket 6 \rrbracket$ be given by the table below.

x	0	1	2	3	4	5
$g(x)$	2	0	4	3	5	1

- (b) Give the table of $g^{(2)}$. [2]
- (c) List the fixed points of g . [1]
- (d) Find $O_g(0)$ and $O_g(3)$. [1]

4

6 marks

Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be given by $h(n) = n/2$ if n is even and $h(n) = 3n + 1$ if n is odd.

- (a) List the iterates $h^{(n)}(6)$ until they repeat, and write down $O_h(6)$. [3]
- (b) Is 1 a fixed point of h ? Find $O_h(1)$. [1]
- (c) Let $k: \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}$ send x to the element of $\mathbb{Z}_{10}$ congruent to x^2 modulo 10. Find $O_k(2)$ and the fixed points of k . [2]

5

6 marks

Find the least element of each set, which exists by the well-ordering principle.

- (a) $\{n \in \mathbb{N} \mid n^2 > 50\}$ [1]
- (b) $\{n \in \mathbb{N} \mid n > 1 \text{ and } n \text{ divides } 91\}$ [1]
- (c) $\{n \in \mathbb{N} \mid 2^n > 1000\}$ [1]
- (d) $\{n \in \mathbb{N} \mid n \equiv 3 \pmod{7} \text{ and } n \equiv 1 \pmod{4}\}$ [1]
- (e) The set $\{x \in \mathbb{Q} \mid x > 0\}$ has no least element. Why does this not contradict the well-ordering principle? [2]

6

6 marks

- (a) Write 360 as a product of primes. [1]
- (b) Write 1001 as a product of primes. [1]
- (c) Write $2^{10} - 1$ as a product of primes. [2]
- (d) Find the least element of $\{k \in \mathbb{N} \mid k > 1 \text{ and } k \text{ divides } 899\}$, and check that it is prime. [2]

7

6 marks

This question is about the Fibonacci numbers $f(n)$.

- (a) Find $f(13)$, $f(14)$ and $f(15)$. [1]
- (b) For which $n \leq 15$ is $f(n)$ even? [2]
- (c) Find $f(n)$ modulo 3 for $n = 0, 1, \dots, 9$. [2]
- (d) Guess for which n the number $f(n)$ is a multiple of 3. [1]

8

6 marks

Let $t(n)$ be the number of strings of length n over $\{a, b, c\}$ with no two neighbouring letters both equal to a .

- (a) Find $t(1)$ and $t(2)$. [1]
- (b) Find $t(3)$, by counting the strings of length 3 that do contain aa . [2]
- (c) By looking at how a string starts, explain why $t(n+2) = 2t(n+1) + 2t(n)$. [2]
- (d) Use this to find $t(4)$. [1]

9

7 marks

- (a) How many people must be in a room to be sure two share a birth month? [1]
- (b) A drawer has socks of 5 colours. How many socks must you take to be sure of a matching pair? [1]
- (c) How many numbers must you choose from $\mathbb{Z}_{20}$ to be sure two are congruent modulo 6? [2]
- (d) Using residues as pigeonholes, explain why among any 6 integers there are two whose difference is a multiple of 5. [3]

10

7 marks

For a function on a finite set, injective and surjective are equivalent. Let $f, g: \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$ send x to the element of $\mathbb{Z}_5$ congruent to $2x$, respectively x^2 , modulo 5.

- (a) Give the table of f . Is f injective? Surjective? [2]
- (b) Give the table of g . Is g injective? Surjective? Justify both answers. [3]
- (c) How many injections are there from $\mathbb{Z}_5$ to $\mathbb{Z}_4$? [1]
- (d) How many surjections are there from $\mathbb{Z}_4$ to $\mathbb{Z}_5$? [1]

11

9 marks

Exam style. Let $f(0) = 0$ and $f(n+1) = f(n) + 3n + 1$.

- (a) Find $f(1), \dots, f(5)$. [2]
- (b) Find the differences $f(n+1) - f(n)$ for $n = 0, \dots, 4$. [1]
- (c) Check that $p(n) = n(3n-1)/2$ agrees with $f(n)$ for $n = 0, \dots, 5$. [2]
- (d) Check the algebra needed for the induction step: show $p(k) + 3k + 1 = p(k+1)$. [3]
- (e) Find $f(20)$. [1]

12

9 marks

Exam style.

- (a) How many functions $\mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}$ are injective? How many are surjective? [2]
- (b) Explain why any 11 numbers chosen from $\{1, 2, \dots, 20\}$ include two that differ by exactly 1. [3]
- (c) Give 10 numbers from $\{1, 2, \dots, 20\}$, no two differing by 1. [1]
- (d) The orbits of the permutation g in question 3 form a partition of $\mathbb{Z}_6$. Write it in bar notation. [2]
- (e) Is 0 a fixed point of $g^{(5)}$? [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.