

Topic 11 · Sheet A: Worked answers and mark scheme

Induction, recursion and the pigeonhole principle

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 10

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Computing recursively

4 marks

(a) 2, 5, 14, 41, since $3 \cdot 1 - 1 = 2$, $3 \cdot 2 - 1 = 5$, and so on. (b) 0, 1, 3, 6: $g(1) = g(0) + 0$, $g(2) = g(1) + 1$, $g(3) = g(2) + 2$, $g(4) = g(3) + 3$.

B2 (a) all four. *B1 for two or three* 2

B2 (b) all four. *B1 for two or three* 2

Common mistake In (b), using $g(n+1) = g(n) + (n+1)$. The rule adds n , the *old* index, so $g(1) = 0$.

2 Partial sums

4 marks

(a) $1 + 3 + 5 + 7 + 9 = 25$. (b) $0 + 1 + 4 + 9 + 16 + 25 = 55$. (c) $1 + 2 + 4 + 8 + 16 = 31$, and the guess is $S(n) = 2^n - 1$.

B1	(a)	1
B1	(b)	1
B1	(c) 31	1
B1	(c) $2^n - 1$	1

Why this question Part (a) is the classic result again: the sum of the first 5 odd numbers is 5^2 . Part (c) shows the usual first step of an induction: compute, then guess.

3 Iterates, fixed points and orbits

6 marks

- (a) $f^{(n)}(x) = 3^n x$, so $f^{(3)}(2) = 54$.
(b) $g^{(2)}(x) = g(g(x))$: 4, 2, 5, 3, 1, 0 for $x = 0, \dots, 5$.
(c) **Only 3**.
(d) $0 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 1 \rightarrow 0$, so $O_g(0) = \{0, 1, 2, 4, 5\}$ and $O_g(3) = \{3\}$.

B1	(a) formula	1
B1	(a) 54	1
M1	(b) applying g twice	1
A1	(b) all six values	1
B1	(c)	1
B1	(d) both orbits	1

Why this question Since g is a permutation, its orbits split $\llbracket 6 \rrbracket$ into pieces. Question 12 uses this.

4 Orbits that are not permutations

6 marks

(a) $6, 3, 10, 5, 16, 8, 4, 2, 1, 4, \dots$, so $O_h(6) = \{1, 2, 3, 4, 5, 6, 8, 10, 16\}$.

(b) **No**: $h(1) = 4$. $O_h(1) = \{1, 2, 4\}$.

(c) $2 \rightarrow 4 \rightarrow 6 \rightarrow 6$, so $O_k(2) = \{2, 4, 6\}$. Fixed points: $0, 1, 5, 6$.

M1	(a) iterating correctly	1
A1	(a) the sequence until it repeats	1
A1	(a) the orbit as a set	1
B1	(b)	1
B1	(c) orbit	1
B1	(c) fixed points	1

Why this question For a function that is not a permutation, an orbit can lead into a cycle without returning to its start: 6 is never reached again. Nobody knows whether every orbit of h reaches 1; this is the Collatz problem.

5 Least elements

6 marks

(a) **8**: $7^2 = 49$, $8^2 = 64$. (b) **7**: $91 = 7 \times 13$. (c) **10**: $2^9 = 512$, $2^{10} = 1024$. (d) **17**: the numbers $\equiv 3 \pmod{7}$ are $3, 10, 17, \dots$, and 17 is the first $\equiv 1 \pmod{4}$.

(e) The well-ordering principle is about **subsets of $\mathbb{N}$** . This set lies in $\mathbb{Q}$, and for any $x > 0$ the smaller number $x/2$ is also in it.

B1	each of (a) to (d)	4
E1	(e) the theorem only applies to subsets of $\mathbb{N}$	1
E1	(e) $x/2$ shows there is no least element	1

Why this question The well-ordering principle is special to $\mathbb{N}$. It is what makes “take the smallest counterexample” a valid move in proofs.

6 Prime factorisations

6 marks

(a) $2^3 \times 3^2 \times 5$. (b) $7 \times 11 \times 13$. (c) $1023 = 3 \times 11 \times 31$. (d) Testing $2, 3, 5, \dots$: the first divisor is **29**, since $899 = 29 \times 31$. It is prime: none of $2, 3, 5$ divides it, and $\sqrt{29} < 6$.

B1	(a)	1
B1	(b)	1
M1	(c) a first factor, such as 3	1
A1	(c)	1
B1	(d) 29	1
B1	(d) primality check	1

Why this question Part (d) shows why every number above one has a prime factor: the least divisor bigger than one is always prime, because any smaller factor would also be a divisor.

7 Fibonacci patterns

6 marks

- (a) 233, 377, 610.
 (b) $n = 0, 3, 6, 9, 12, 15$: the even values are 0, 2, 8, 34, 144, 610.
 (c) 0, 1, 1, 2, 0, 2, 2, 1, 0, 1.
 (d) **When n is a multiple of 4**: zeros appear at $n = 0, 4, 8$, and the pattern repeats every 8 steps.

B1	(a)	1
B2	(b) all six. B1 for four or five	2
B2	(c) all ten. B1 for eight or nine	2
B1	(d)	1

Why this question Part (b) is a repeating pattern. Working modulo 2 or 3 shows why: each value depends only on the previous two, so once a pair repeats, the whole sequence repeats.

8 A recursion for counting strings

6 marks

- (a) $t(1) = 3$; $t(2) = 9 - 1 = 8$, removing aa .
 (b) Strings containing aa : $aa*$ (3), $*aa$ (3), with aaa counted twice, so 5. Thus $t(3) = 27 - 5 = 22$.
 (c) A valid string of length $n + 2$ either starts with b or c , followed by any valid string of length $n + 1$ (giving $2t(n + 1)$), or starts with a , which must be followed by b or c and then any valid string of length n (giving $2t(n)$).
 (d) $t(4) = 2 \times 22 + 2 \times 8 = 60$.

B1	(a) both	1
M1	(b) counting by complement, avoiding the double count	1
A1	(b) 22	1
E1	(c) case where it starts with b or c	1
E1	(c) case where it starts with a	1
B1	(d) 60	1

Why this question This is strong recursion: each value needs the two before it, as with Fibonacci. Check: $t(2) = 2t(1) + 2t(0) = 6 + 2 = 8$, with $t(0) = 1$ for the empty string.

9 The pigeonhole principle

7 marks

- (a) **13**. (b) **6**. (c) There are 6 residues modulo 6, and all six occur in $\llbracket 20 \rrbracket$, so **7**.
(d) There are 5 residues modulo 5. Six integers in five residue classes means two share a class, and then their difference is a multiple of 5.

B1	(a)	1
B1	(b)	1
M1	(c) residues as pigeonholes	1
A1	(c) 7	1
E1	(d) five residues modulo 5 as pigeonholes	1
E1	(d) six integers, so two share a residue	1
E1	(d) same residue gives difference a multiple of 5	1

Why this question The trick is always choosing the pigeonholes. Residues modulo n are a natural choice, because “same residue” means “difference divisible by n ”.

10 Injective versus surjective on finite sets

7 marks

- (a) **0, 2, 4, 1, 3**. **Injective and surjective**.
(b) **0, 1, 4, 4, 1**. **Neither**: $g(2) = g(3)$, and 2 is missed.
(c) **0**, by the pigeonhole principle.
(d) **0**: four inputs cannot hit five values.

B1	(a) table	1
B1	(a) both properties	1
B1	(b) table	1
B1	(b) not injective: $g(2) = g(3)$	1
B1	(b) not surjective: 2 is missed	1
B1	(c)	1
B1	(d)	1

Why this question In each of (a) and (b) the two properties stand or fall together, as they always do on a finite set. This fails on infinite sets: $n \mapsto 2n$ on $\mathbb{N}$ is injective but not surjective.

11 Exam style: guessing a formula

9 marks

- (a) **1, 5, 12, 22, 35**.
(b) **1, 4, 7, 10, 13**.
(c) $p(0), \dots, p(5)$ are 0, 1, 5, 12, 22, 35: **they agree**.

$$(d) \ p(k) + 3k + 1 = \frac{3k^2 - k + 6k + 2}{2} = \frac{3k^2 + 5k + 2}{2} = \frac{(k+1)(3k+2)}{2} = p(k+1). \text{ It checks.}$$

$$(e) \ f(20) = p(20) = 20 \times 59/2 = 590.$$

B2	(a) all five. B1 for three or four	2
B1	(b)	1
M1	(c) evaluating p	1
A1	(c) all six agree	1
M1	(d) common denominator	1
M1	(d) factorising $3k^2 + 5k + 2 = (k+1)(3k+2)$	1
A1	(d) reaching $p(k+1)$	1
B1	(e)	1

Why this question Parts (c) and (d) are the two halves of an induction: base case and induction step. Here the base case is $f(0) = 0 = p(0)$, and (d) is exactly what the step needs.

12 Exam style: counting and pigeonholes

9 marks

- (a) $10! = 3628800$ are injective, and **the same number** are surjective, since on a finite set the two properties agree.
- (b) Pair up $\{1,2\}, \{3,4\}, \dots, \{19,20\}$: ten pigeonholes. Eleven numbers means two land in one pair, and those two differ by 1.
- (c) For example $1,3,5,\dots,19$.
- (d) $01245|3$.
- (e) **Yes**: the orbit of 0 is a cycle of length 5, so five steps return to 0.

B1	(a) injective count	1
B1	(a) surjective count, same	1
M1	(b) the ten pairs as pigeonholes	1
E1	(b) eleven numbers in ten pairs, so two share a pair	1
E1	(b) numbers in the same pair differ by 1	1
B1	(c)	1
M1	(d) using the orbits from question 3	1
A1	(d)	1
B1	(e)	1

Why this question Part (c) shows (b) is as strong as it can be: with only 10 numbers it can fail. Pigeonhole bounds are often exactly sharp like this.

Total: 76 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.