

Topic 10 · Sheet A: Questions

Proof techniques: direct, contrapositive, contradiction, construction

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 9

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. This sheet is about the *tools* of proof: implications, converses, contrapositives, counterexamples, cases, and checking candidates by construction. You will not be asked to write full proofs. You may use anything from Topics 1 to 9.

Notation. For “if P then Q ”: the converse is “if Q then P ” and the contrapositive is “if not Q then not P ”. A prime is a natural number greater than one with exactly two positive divisors. An integer k is a sum of two squares if $k = x^2 + y^2$ for some integers x and y ; 0 counts as a square.

1

4 marks

Consider the statement: “If n is an even integer then n^2 is an even integer.”

- (a) State its hypothesis and its conclusion. [1]
- (b) Write down its converse. [1]
- (c) Write down its contrapositive. [1]
- (d) Which of the statement, its converse and its contrapositive hold? [1]

2

4 marks

Consider the statement about real numbers: “If $x^2 > 4$ then $x > 2$.”

- (a) Does it hold? If not, give a counterexample. [1]
- (b) Write down its converse. Does the converse hold? [1]
- (c) Write down its contrapositive. Does it hold? [1]
- (d) Which of the two, converse or contrapositive, always has the same truth value as the original? [1]

3

6 marks

Find a counterexample to each statement.

- (a) Every odd natural number greater than 1 is prime. [1]
- (b) For every integer n , if $n^2 \equiv 1 \pmod{8}$ then $n \equiv 1 \pmod{8}$. [1]
- (c) For every $n \in \mathbb{N}$, the number $n^2 + n + 41$ is prime. [2]
- (d) For all sets X, Y, Z : if $X \cup Y = X \cup Z$ then $Y = Z$. [2]

4

6 marks

This question works through the squares modulo 4, case by case.

- (a) Find x^2 modulo 4 for each $x \in \llbracket 4 \rrbracket$. [1]
- (b) Which values can $x^2 + y^2$ take modulo 4? [1]
- (c) For each of 19, 25, 27 and 34, decide whether it is a sum of two squares. Give x and y when it is; when it is not, give a reason. [4]

5

6 marks

- (a) Find x^2 modulo 8 for each $x \in \llbracket 8 \rrbracket$, and list the possible values. [2]
- (b) Which values in $\llbracket 8 \rrbracket$ can $x^2 + y^2 + z^2$ take modulo 8? [2]
- (c) Is 15 a sum of three squares? Is 14? [2]

6

6 marks

This question tests numbers for primality by trial division up to $\sqrt{N}$.

- (a) To show that 157 is prime, which primes do you need to test as divisors? [1]
- (b) Find 157 modulo each of those primes, and conclude. [2]
- (c) Is 221 prime? [1]
- (d) Find the smallest prime greater than 300, checking your answer. [2]

7

6 marks

- (a) Find a natural number less than 100 with three different prime factors. [1]
- (b) List every natural number less than 100 with exactly three different prime factors. [3]
- (c) Find the smallest odd natural number with three different prime factors. [2]

8

6 marks

Name the pattern of proof each extract uses: direct, contraposition, contradiction, or construction.

- (a) "Suppose n is odd. Then n^2 is odd. So if n^2 is even, n is even." [1]
- (b) "Suppose, for a contradiction, that $r^2 = 3$ for some rational $r \dots$ " [1]

- (c) "Take $N = 211$. We check that no prime less than 15 divides $N \dots$ " [1]
- (d) "Suppose n is even, say $n = 2k$. Then $n^2 = 4k^2 = 2(2k^2)$." [1]
- (e) Write down the tautologies that justify the patterns in (a) and (b). [2]

9

7 marks

- (a) "An integer n is odd if and only if n^2 is odd." Write this as two implications. [2]
- (b) Write down the contrapositive of each of your two implications. [2]
- (c) Check the equivalence " $X \subset Y$ if and only if $X \cap Y = X$ " when $Y = \{1, 2, 3\}$, first with $X = \{1, 2\}$ and then with $X = \{1, 4\}$. Explain what your checks show. [3]

10

7 marks

- (a) Find x^2 modulo 3 for each $x \in \mathbb{Z}_3$. [1]
- (b) Can a perfect square be congruent to 2 modulo 3? [1]
- (c) Which of 47, 49, 77 and 98 can you rule out as perfect squares using (b)? [2]
- (d) Is the converse of "if $n \equiv 2 \pmod{3}$ then n is not a perfect square" true? State the converse and justify your answer. [3]

11

9 marks

Exam style. Let S be the statement: "For every integer n , if n^2 is a multiple of 3 then n is a multiple of 3."

- (a) Write down the contrapositive of the implication in S . [1]
- (b) Check the contrapositive in the two cases $n \equiv 1 \pmod{3}$ and $n \equiv 2 \pmod{3}$. [3]
- (c) Write down the converse of the implication in S . Does it hold? [2]
- (d) Write the negation of S using quantifiers, with $\neg$ moved inside. [2]
- (e) Does the negation of S hold? [1]

12

9 marks

Exam style.

- (a) List every $n \in \mathbb{Z}_{21}$ that is a sum of two squares. [4]
- (b) Which of the remaining numbers in $\mathbb{Z}_{21}$ are congruent to 3 modulo 4? [1]
- (c) Which numbers in $\mathbb{Z}_{21}$ are neither sums of two squares nor congruent to 3 modulo 4? [2]
- (d) What does (c) tell you about the converse of "if $k \equiv 3 \pmod{4}$ then k is not a sum of two squares"? [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.