

Topic 10 · Sheet A: Worked answers and mark scheme

Proof techniques: direct, contrapositive, contradiction, construction

Suggested time

About 75 minutes

Total marks

76

Questions

12

Builds on

Topics 1 to 9

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Converse and contrapositive

4 marks

(a) Hypothesis " **n is an even integer**"; conclusion " **n^2 is an even integer**".

(b) **If n^2 is even then n is even.**

(c) **If n^2 is not even then n is not even**; that is, if n^2 is odd then n is odd.

(d) **All three hold.** The statement holds, since $(2k)^2 = 2(2k^2)$; the contrapositive always agrees with it; the converse holds because an odd number squared is odd.

B1 each part correct

4

Why this question Here the converse happens to hold too, so the statement is really an equivalence. That is a coincidence of this example, as question 2 shows.

2 When the converse and original differ

4 marks

(a) **No:** $x = -3$ has $x^2 = 9 > 4$ but $x < 2$.

(b) Converse: if $x > 2$ then $x^2 > 4$. **It holds.**

(c) Contrapositive: if $x \leq 2$ then $x^2 \leq 4$. **It fails**, again at $x = -3$.

(d) **The contrapositive.**

B1 each part correct

4

Common mistake Treating the converse as a restatement of the original. It is a different claim, and here it has a different truth value. Only the contrapositive is equivalent, by the tautology $((P \rightarrow Q) \leftrightarrow ((\neg Q) \rightarrow (\neg P)))$.

3 Counterexamples

6 marks

- (a) $9 = 3 \times 3$.
- (b) $n = 3$: $9 \equiv 1 \pmod{8}$ but $3 \not\equiv 1 \pmod{8}$.
- (c) $n = 40$: $40^2 + 40 + 41 = 1681 = 41^2$, which is not prime.
- (d) $X = \{0\}$, $Y = \{0\}$, $Z = \emptyset$: both unions are $\{0\}$, but $Y \neq Z$.

B1	(a)	1
B1	(b)	1
M1	(c) looking for n making the expression a multiple of 41	1
A1	(c) $n = 40$ or $n = 41$, with the factorisation	1
M1	(d) choosing $Y \neq Z$ that differ only inside X	1
A1	(d) a correct example	1

Why this question Part (c) is famous: $n^2 + n + 41$ is prime for $n = 0, 1, \dots, 39$. Forty examples prove nothing; one counterexample settles it.

4 Sums of two squares

6 marks

- (a) $0, 1, 0, 1$.
- (b) Each square is 0 or 1, so the sum is $0, 1$ or 2 , never 3.
- (c) **19: no**, since $19 \equiv 3 \pmod{4}$. **25: yes**, $3^2 + 4^2$ (or $5^2 + 0^2$). **27: no**, since $27 \equiv 3 \pmod{4}$. **34: yes**, $3^2 + 5^2$.

B1	(a)	1
B1	(b)	1
B1	(c) each number with a valid reason or pair	4

Why this question This is the case analysis behind sums of two squares, done as a table. Working modulo 4 reduces infinitely many cases to four.

5 Sums of three squares

6 marks

- (a) 0, 1, 4, 1, 0, 1, 4, 1, so the possible values are 0, 1 and 4.
- (b) Adding three values from $\{0, 1, 4\}$ gives 0, 1, 2, 3, 4, 5, 6, but never 7.
- (c) 15: no, since $15 \equiv 7 \pmod{8}$. 14: yes, $14 = 3^2 + 2^2 + 1^2$.

B1	(a) table	1
B1	(a) values 0, 1, 4	1
M1	(b) adding three values from the list	1
A1	(b) all but 7	1
B1	(c) 15 with reason	1
B1	(c) 14 with an example	1

Why this question This turns a proof by cases into a calculation: the table in (b) is exactly the case analysis a proof would write out.

6 Checking primes by hand

6 marks

- (a) $\sqrt{157} \approx 12.5$, so 2, 3, 5, 7, 11.
- (b) The remainders are 1, 1, 2, 3, 3, none zero, so 157 is prime.
- (c) No: $221 = 13 \times 17$.
- (d) $301 = 7 \times 43$; 302 and 304 and 306 are even; $303 = 3 \times 101$; $305 = 5 \times 61$. For 307, test primes up to 17: remainders are 1, 1, 2, 6, 10, 8, 1. So 307.

B1	(a)	1
M1	(b) all five remainders	1
A1	(b) conclusion	1
B1	(c) with factors	1
M1	(d) ruling out 301 to 306	1
A1	(d) 307 with checks	1

Why this question This explains why primes up to $\sqrt{N}$ are enough: if all prime factors were bigger than $\sqrt{N}$, their product would exceed N .

7 Constructing examples

6 marks

- (a) For example $30 = 2 \times 3 \times 5$.
- (b) 30, 42, 60, 66, 70, 78, 84, 90: eight numbers.

(c) $105 = 3 \times 5 \times 7$.

B1	(a)	1
M1	(b) a systematic search, such as by smallest prime factors	1
A2	(b) all eight. <i>A1 for six or seven, none wrong</i>	2
M1	(c) using the three smallest odd primes	1
A1	(c) 105	1

Common mistake In (b), missing $60 = 2^2 \times 3 \times 5$ or $84 = 2^2 \times 3 \times 7$. “Three different prime factors” allows a prime to appear more than once.

8 Recognising patterns of proof

6 marks

(a) **Contraposition.** (b) **Contradiction.** (c) **Construction.** (d) **Direct.**
 (e) $((P \rightarrow Q) \leftrightarrow ((\neg Q) \rightarrow (\neg P)))$ for (a) and $((\neg P) \rightarrow F) \rightarrow P$ for (b).

B1	each of (a) to (d)	4
B1	(e) contraposition tautology	1
B1	(e) argument by contradiction	1

Why this question Each pattern of proof rests on a tautology. Knowing which one tells you exactly what you are allowed to assume and what you must reach.

9 Splitting an equivalence

7 marks

- (a) **If n is odd then n^2 is odd, and if n^2 is odd then n is odd.**
 (b) **If n^2 is even then n is even, and if n is even then n^2 is even.**
 (c) With $X = \{1, 2\}$: $X \subset Y$ is true and $X \cap Y = \{1, 2\} = X$ is true. With $X = \{1, 4\}$: $X \subset Y$ is false and $X \cap Y = \{1\} \neq X$. **Both sides agree each time.**

B1	(a) each implication	2
B1	(b) each contrapositive	2
B1	(c) first check	1
B1	(c) second check	1
E1	(c) the two sides have the same truth value each time	1

Why this question Notice that the contrapositives in (b) are the facts about odd squares and their converses. Proving an “if and only if” means proving two implications, and either can be done by contraposition.

10 Ruling out squares

7 marks

- (a) 0, 1, 1.
- (b) **No**: squares are 0 or 1 modulo 3.
- (c) 47, 49, 77, 98 are 2, 1, 2, 2 modulo 3. Rule out 47, 77 and 98. The test says nothing about 49, which is in fact 7^2 .
- (d) The converse says: if n is not a perfect square then $n \equiv 2 \pmod{3}$. **False**: 3 is not a square, and $3 \equiv 0$.

B1	(a)	1
B1	(b)	1
B2	(c) exactly 47, 77 and 98. B1 for two of them	2
B1	(d) converse stated	1
B1	(d) false	1
E1	(d) a counterexample such as 3	1

Why this question A modular test can only rule things out. Part (c) shows it cannot confirm that a number *is* a square, which is the converse failing in practice.

11 Exam style: multiples of 3

9 marks

- (a) **If n is not a multiple of 3 then n^2 is not a multiple of 3.**
- (b) If $n \equiv 1$ then $n^2 \equiv 1$. If $n \equiv 2$ then $n^2 \equiv 4 \equiv 1 \pmod{3}$, so it is not a multiple of 3.
- (c) **If n is a multiple of 3 then n^2 is a multiple of 3. It holds: $n = 3k$ gives $n^2 = 3(3k^2)$.**
- (d) **$(\exists n ((n^2 \text{ is a multiple of } 3) \wedge (n \text{ is not a multiple of } 3)))$.**
- (e) **No**: part (b) shows no such n exists.

B1	(a)	1
M1	(b) working modulo 3 in both cases	1
A1	(b) $n \equiv 1$ gives $n^2 \equiv 1$	1
A1	(b) $n \equiv 2$ gives $n^2 \equiv 1$	1
B1	(c) converse	1
B1	(c) holds, with reason	1
M1	(d) $\exists$ with $\wedge$	1
A1	(d) correct negation	1
B1	(e)	1

Why this question This is the key step in proving that $\sqrt{3}$ is irrational, in the same way the fact about even squares is used for $\sqrt{2}$.

12 Exam style: testing a converse

9 marks

- (a) 0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20.
- (b) 3, 7, 11, 15, 19.
- (c) 6, 12 and 14.
- (d) The converse of “if $p \equiv 3 \pmod{4}$ then p is not a sum of two squares” would say: if p is not a sum of two squares then $p \equiv 3 \pmod{4}$. **It is false**; 6 is a counterexample.

M1	(a) a systematic search over $x \leq y$	1
A3	(a) all thirteen. <i>A2 for eleven or twelve, A1 for nine or ten, none wrong</i>	3
B1	(b)	1
B2	(c) all three. <i>B1 for two</i>	2
B1	(d) the converse stated	1
E1	(d) false, with a counterexample	1

Why this question Being 3 modulo 4 gives a reason some numbers are not sums of two squares, but not the only reason. The numbers 6, 12 and 14 fail for a deeper reason, involving their prime factors.

Total: 76 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.