

Topic 9 · Sheet A: Questions

Logic: booleans, truth tables and quantifiers

Suggested time

About 75 minutes

Total marks

75

Questions

12

Builds on

Topics 1 to 8

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need booleans and the basic operators, boolean algebra, tautologies, truth tables and quantifiers, plus Topics 1 to 8.

Notation. $\mathbb{B} = \{T, F\}$. The basic operators are $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$, and expressions are fully bracketed. Truth tables list rows in lexicographic order: TT, TF, FT, FF for two variables. A tautology is always T; an antinomy is always F. $\text{Prime}(n) = T$ exactly when n is prime. Unless stated otherwise, quantifiers range over $\mathbb{N}$.

1

4 marks

Evaluate each expression.

- (a) $(T \wedge F)$ [1]
- (b) $(F \rightarrow T)$ [1]
- (c) $(\neg(T \leftrightarrow F))$ [1]
- (d) $((F \vee T) \rightarrow F)$ [1]

2

4 marks

- (a) List the strings of length 2 over $\{T, F\}$ in lexicographic order. [1]
- (b) Which comes first in lexicographic order: 01101 or 01011? [1]
- (c) List, in order, the binary strings u of length 4 with $0110 \leq u \leq 1001$. [2]

3

6 marks

- (a) Use a truth table, clause by clause, to find the lookup table of $((P \rightarrow Q) \wedge (Q \rightarrow P))$. [3]
- (b) Which basic operator has the same lookup table? [1]
- (c) Find the lookup table of $((\neg P) \vee Q)$, and say which basic operator it equals. [2]

4

6 marks

- (a) How many boolean operators are there of arity 1? Of arity 2? Of arity 3? [3]
- (b) How many rows does the lookup table of an operator of arity 4 have? [1]
- (c) List the clauses of $((P \wedge Q) \vee (\neg R))$ and give the length of each, counting every symbol including brackets. [2]

5

6 marks

Decide whether each is a tautology, an antinomy, or neither.

- (a) $(P \vee (\neg P))$ [1]
- (b) $(P \wedge (\neg P))$ [1]
- (c) $(P \rightarrow (Q \rightarrow P))$ [1]
- (d) $((P \rightarrow Q) \rightarrow P)$ [1]
- (e) $((P \wedge Q) \rightarrow (P \vee Q))$ [1]
- (f) $((P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow (\neg Q)))$ [1]

6

6 marks

Simplify each expression as far as possible, using the laws of boolean algebra or a truth table.

- (a) $(P \vee F)$ [1]
- (b) $(P \wedge (P \vee Q))$ [1]
- (c) $(\neg((\neg P) \wedge Q))$, to an expression with no $\wedge$ [1]
- (d) $((P \wedge Q) \vee (P \wedge (\neg Q)))$ [1]
- (e) $((\neg(\neg P)) \vee P)$ [1]
- (f) $(P \rightarrow F)$ [1]

7

6 marks

Write each expression using only $\neg$ and $\vee$. Check your answer with a truth table.

- (a) $(P \rightarrow (\neg Q))$ [2]
- (b) $((\neg P) \wedge Q)$ [2]
- (c) $(P \wedge (\neg Q))$ [2]

8

6 marks

- (a) Find the lookup table of $f = ((P \wedge Q) \rightarrow R)$, with the eight rows in lexicographic order. [3]
- (b) In how many rows is f true? [1]
- (c) Is f equal, as a function, to $(P \rightarrow (Q \rightarrow R))$? [2]

9

6 marks

Decide whether each sentence holds.

- (a) $(\forall n (n \geq 0))$ [1]
- (b) $(\exists n (n^2 = 2))$ [1]
- (c) $(\forall n (\exists m (m > n)))$ [1]
- (d) $(\exists m (\forall n (m > n)))$ [1]
- (e) $(\forall n (\exists m (n = 2m)))$ [1]
- (f) $(\exists n (\text{Prime}(n) \wedge (n \text{ is even})))$ [1]

10

7 marks

Use the rules for negating quantifiers to write the negation of each sentence with $\neg$ moved inside the quantifiers. Then decide whether the original sentence holds.

- (a) $(\forall n (n^2 \geq n))$ [2]
- (b) $(\exists n (\forall m (n \leq m)))$ [2]
- (c) $(\forall n (\text{Prime}(n) \rightarrow (n \text{ is odd})))$, and give the n that settles it [3]

11

9 marks

Exam style. Let $f = ((P \vee Q) \rightarrow (\neg(P \wedge Q)))$.

- (a) Find the lookup table of f clause by clause, showing a column for every clause. [4]
- (b) Find a shorter expression of f using one $\neg$ and one $\wedge$. [2]
- (c) Write f using only $\neg$ and $\vee$. [1]
- (d) Is $(f \leftrightarrow ((\neg P) \vee (\neg Q)))$ a tautology? Explain using lookup tables. [2]

12

9 marks

Exam style. In parts (a) to (e) the background set is $\mathbb{Z}$.

- (a) Does $(\forall x (\exists y (x + y = 2)))$ hold? [1]
- (b) Does $(\exists y (\forall x (x + y = 2)))$ hold? [1]
- (c) Does $(\forall x (\exists y (x < y)))$ hold? [1]
- (d) Does $(\exists x (\forall y (x \leq y)))$ hold? [1]
- (e) Write the negation of (c) with $\neg$ moved inside, and give an x that witnesses it. [2]
- (f) Over $\mathbb{N}$, let $R(x)$ say x is even and $S(x)$ say x is odd. Decide whether $(\forall x (R(x) \vee S(x)))$ and $((\forall x R(x)) \vee (\forall x S(x)))$ hold. What does this show? [3]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.