

Topic 9 · Sheet A: Worked answers and mark scheme

Logic: booleans, truth tables and quantifiers

Suggested time

About 75 minutes

Total marks

75

Questions

12

Builds on

Topics 1 to 8

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Evaluating expressions

4 marks

(a) **F** (b) **T**: an implication with false hypothesis is true. (c) $(T \leftrightarrow F) = F$, so **T**. (d) $(F \vee T) = T$ and $(T \rightarrow F) = F$: **F**.

B1 each part correct

4

Why this question Part (b) is the row of the truth table for $\rightarrow$ that surprises most people. "If F then anything" is counted as true.

2 Lexicographic order

4 marks

(a) TT, TF, FT, FF. (b) 01011: the first difference is in position 2, where it has 0. (c) 0110, 0111, 1000, 1001.

B1	(a)	1
B1	(b)	1
B2	(c) all four in order. B1 for three, or four out of order	2

Why this question Lexicographic order is dictionary order, and it fixes the order of rows in every truth table, so two people's tables can be compared line by line.

3 Truth tables

6 marks

(a) Working clause by clause:

P	Q	$(P \rightarrow Q)$	$(Q \rightarrow P)$	$((P \rightarrow Q) \wedge (Q \rightarrow P))$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(b) $\leftrightarrow$.

(c) Values T, F, T, T, which is $(P \rightarrow Q)$.

M1	(a) both inner clauses evaluated	1
A2	(a) final column. A1 for three rows correct	2
B1	(b)	1
B1	(c) lookup table	1
B1	(c) $\rightarrow$	1

Why this question These are the standard tautologies linking equivalence and implication, checked rather than taken on trust.

4 Counting operators and clauses

6 marks

(a) An operator of arity n is a function $\mathbb{B}^n \rightarrow \mathbb{B}$, and $\mathbb{B}^n$ has 2^n elements. So there are $2^2 = 4$, $2^4 = 16$ and $2^8 = 256$.

(b) $2^4 = 16$.

(c) Three clauses: $(P \wedge Q)$ of length 5, $(\neg R)$ of length 4, and the whole expression, of length 12.

B1	(a) each count	3
B1	(b) 16	1
B1	(c) the three clauses	1
B1	(c) the three lengths	1

Why this question Part (a) is counting functions again (Topic 2): $2^{(2^n)}$ operators of arity n . With 16 of arity 2 and only 4 basic ones, twelve more exist.

5 Tautologies and antinomies

6 marks

(a) **Tautology** (excluded middle). (b) **Antinomy**. (c) **Tautology**: if $P = T$ then $(Q \rightarrow T) = T$; if $P = F$ the implication is true. (d) **Neither**: its lookup table is T, T, F, F, the same as P . (e) **Tautology**. (f) **Antinomy**: $(P \leftrightarrow Q)$ and $(P \leftrightarrow (\neg Q))$ always have opposite values.

B1	each part correct	6
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Common mistake In (d), assuming it is a tautology because it looks like modus ponens. Only a full truth table settles it: row FT gives $(T \rightarrow F) = F$.

6 Simplifying

6 marks

(a) P (identity law). (b) P (absorption). (c) By De Morgan, $((\neg(\neg P)) \vee (\neg Q))$, which is $(P \vee (\neg Q))$. (d) By distributivity, $(P \wedge (Q \vee (\neg Q))) = (P \wedge T) = P$. (e) P . (f) $(\neg P)$.

B1	each part correct	6
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Why this question You can always replace a clause by an equivalent one. Each answer here is one or two such replacements, and a truth table always confirms it.

7 Using only $\neg$ and $\vee$

6 marks

- (a) $(A \rightarrow B) = ((\neg A) \vee B)$, so $((\neg P) \vee (\neg Q))$.
 (b) $(A \wedge B) = (\neg((\neg A) \vee (\neg B)))$, so $(\neg(P \vee (\neg Q)))$, after removing the double negation.
 (c) $(\neg((\neg P) \vee Q))$.

Truth tables, rows TT, TF, FT, FF: (a) F, T, T, T; (b) F, F, T, F; (c) F, T, F, F. Each matches the original expression.

M1	each part, a correct rewriting rule	3
A1	each part, correct answer with a check	3

Why this question Two operators, $\neg$ and $\vee$, are enough to write anything. Doing it by hand shows the price: expressions get longer and harder to read.

8 Three variables

6 marks

- (a) Rows TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF give **T, F, T, T, T, T, T, T**. Only TTF is false, since that is the only row where $(P \wedge Q) = T$ and $R = F$.
- (b) **7**.
- (c) **Yes**. $(P \rightarrow (Q \rightarrow R))$ is false only when $P = T$ and $(Q \rightarrow R) = F$, that is in row TTF. Same lookup table.

M1	(a) eight rows in lexicographic order	1
A2	(a) all values. <i>A1 for seven correct</i>	2
B1	(b) 7	1
M1	(c) comparing lookup tables	1
A1	(c) yes	1

Why this question Finding the single false row is often quicker than filling the whole table. An implication is false in exactly one situation: true hypothesis, false conclusion.

9 Reading quantified sentences

6 marks

- (a) **Holds**. (b) **Does not hold**: no natural number squares to 2. (c) **Holds**: take $m = n + 1$. (d) **Does not hold**: $n = m$ fails $m > n$. (e) **Does not hold**: $n = 1$ is not $2m$ for any m . (f) **Holds**: $n = 2$.

B1	each part correct	6
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Why this question Parts (c) and (d) use the same property with the quantifiers swapped. In (c), m may depend on n ; in (d), one m must work for every n . This is why the order of quantifiers matters.

10 Negating quantifiers

7 marks

- (a) Negation: $(\exists n (n^2 < n))$. The original **holds**, since $n^2 \geq n$ for every natural number.
- (b) Negation: $(\forall n (\exists m (n > m)))$. The original **holds**, with $n = 0$.
- (c) Negation: $(\exists n (\text{Prime}(n) \wedge (n \text{ is not odd})))$. The original **does not hold**: $n = 2$ is prime and even.

M1	(a), (b) swapping $\forall$ and $\exists$ and pushing $\neg$ inside	2
A1	(a), (b) correct truth value	2
M1	(c) negation, with $\rightarrow$ becoming $\wedge \neg$	1
A1	(c) does not hold	1
B1	(c) the witness $n = 2$	1

Common mistake In (c), negating the implication $(A \rightarrow B)$ as $(A \rightarrow (\neg B))$. Its negation is $(A \wedge (\neg B))$, by the definition of implication and De Morgan.

11 Exam style: simplifying an operator

9 marks

(a)

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$(\neg(P \wedge Q))$	f
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	T

(b) The lookup table F, T, T, T is that of $(\neg(P \wedge Q))$.

(c) By De Morgan, $((\neg P) \vee (\neg Q))$.

(d) **Yes**. Both sides have the same lookup table, so the equivalence is a tautology.

M1	(a) inner clauses evaluated	1
A1	(a) the column for $(\neg(P \wedge Q))$	1
A2	(a) final column. <i>A1 for three rows</i>	2
B2	(b)	2
B1	(c)	1
B1	(d) yes	1
E1	(d) same lookup tables, so always T	1

Why this question The row FF is the interesting one: f is true there because the hypothesis $(P \vee Q)$ is false. That is why f simplifies to “not both”.

12 Exam style: the order of quantifiers

9 marks

(a) **Holds**: take $y = 2 - x$, which lies in $\llbracket 3 \rrbracket$.

(b) **Does not hold**: y would have to equal $2 - x$ for every x at once.

(c) **Does not hold**: $x = 2$ has no larger y in $\llbracket 3 \rrbracket$.

(d) **Holds**: $x = 0$.

(e) $(\exists x (\forall y (\neg(x < y))))$, witnessed by $x = 2$.

(f) The first **holds**, since every natural number is even or odd. The second **does not**, since not every number is even and not every number is odd. So $\forall$ **does not distribute over** $\vee$.

B1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e) negation	1
B1	(e) witness $x = 2$	1
B1	(f) the first holds	1
B1	(f) the second does not, with reason	1
E1	(f) the conclusion about distributing	1

Why this question Part (f) shows that $\forall$ distributes over $\wedge$, but not over $\vee$.

Total: 75 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.