

Topic 7 · Sheet A: Questions

Partitions, equivalence classes and the integers

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 6

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need partitions, equivalence classes, quotients, the construction of $\mathbb{Z}$ and well-defined functions, plus Topics 1 to 6.

Notation. Partitions are written with bars, so $01|2$ is $\{\{0,1\}, \{2\}\}$. For an equivalence relation E on X , $[x]_E = \{y \in X \mid xEy\}$, X/E is the set of classes, and $q_E(x) = [x]_E$. On $\mathbb{N}^2$, $(p,q)E(r,s)$ when $p+s = r+q$; then $\mathbb{Z} = \mathbb{N}^2/E$, $k_{\mathbb{Z}} = [(k,0)]_E$ and $-k_{\mathbb{Z}} = [(0,k)]_E$. The operations are $[(p,q)] +_{\mathbb{Z}} [(r,s)] = [(p+r, q+s)]$, $[(p,q)] \times_{\mathbb{Z}} [(r,s)] = [(pr+qs, ps+qr)]$ and $-[(p,q)] = [(q,p)]$, dropping the subscript E .

1

4 marks

Decide whether each is a partition of $\llbracket 4 \rrbracket$. If not, say which condition fails.

- (a) $\{\{0,1\}, \{2,3\}\}$ [1]
- (b) $\{\{0,1\}, \{1,2\}, \{3\}\}$ [1]
- (c) $\{\{0\}, \{1,2\}, \emptyset, \{3\}\}$ [1]
- (d) $\{\{0,2\}, \{3\}\}$ [1]

2

4 marks

- (a) Write $\{\{0,3\}, \{1\}, \{2,4\}\}$ in bar notation. [1]
- (b) Write the partition $0|12|34$ of $\llbracket 5 \rrbracket$ as a set of sets. [1]
- (c) Are $01|23$ and $10|32$ the same partition of $\llbracket 4 \rrbracket$? [1]
- (d) Are $01|23$ and $02|13$ the same partition of $\llbracket 4 \rrbracket$? [1]

3

6 marks

- (a) List every partition of $\llbracket 4 \rrbracket$ with exactly two parts. [3]
- (b) How many partitions of $\llbracket 4 \rrbracket$ have exactly three parts? [2]
- (c) How many partitions of $\llbracket 4 \rrbracket$ have exactly one part? [1]

4

6 marks

On $X = \llbracket 8 \rrbracket$, let xEy when $|x - y|$ is a multiple of 3. You may assume E is an equivalence relation.

- (a) Find $[0]_E$ and $[4]_E$. [2]
- (b) Write X/E in bar notation. [1]
- (c) Find $|X/E|$. [1]
- (d) Find $q_E(7)$, and the set of $x \in X$ with $q_E(x) = [2]_E$. [2]

5

6 marks

Let Q be the equivalence relation on $\llbracket 6 \rrbracket$ whose classes are the parts of $0|12|345$, so xQy when x and y lie in the same part.

- (a) Is $1Q2$? Is $2Q3$? [1]
- (b) How many ordered pairs are in Q ? [2]
- (c) Find $[3]_Q$. [1]
- (d) How many ordered pairs are in the relation coming from the partition into singletons? From the partition with one part? [2]

6

4 marks

Use the fact that equivalence relations match partitions to count.

- (a) How many equivalence relations on $\llbracket 4 \rrbracket$ have exactly two classes? [2]
- (b) How many of these have two classes of size 2? [1]
- (c) How many equivalence relations on $\llbracket 5 \rrbracket$ have exactly four classes? [1]

7

6 marks

This question uses the relation E on $\mathbb{N}^2$ that defines $\mathbb{Z}$.

- (a) Is $(5, 2)E(7, 4)$? [1]
- (b) Is $(3, 8)E(0, 5)$? [1]
- (c) List four elements of $[(2, 0)]_E$. [1]
- (d) Write $[(4, 9)]_E$ as $k_{\mathbb{Z}}$ or $-k_{\mathbb{Z}}$. [1]
- (e) Find the element (p, q) of $[(0, 4)]_E$ with $p + q = 10$. [2]

8

6 marks

Compute each, using the definitions of $+_{\mathbb{Z}}$ and $\times_{\mathbb{Z}}$, and write the answer as $k_{\mathbb{Z}}$ or $-k_{\mathbb{Z}}$.

- (a) $[(5, 2)] +_{\mathbb{Z}} [(1, 4)]$ [2]
- (b) $[(5, 2)] \times_{\mathbb{Z}} [(1, 4)]$ [2]
- (c) $-[(5, 2)]$ [1]

- (d) $3_{\mathbb{Z}} \times_{\mathbb{Z}} (-2_{\mathbb{Z}})$, using the representatives $(3, 0)$ and $(0, 2)$ [1]

9

7 marks

This question checks well-definedness by hand. The pairs $(5, 3)$ and $(9, 7)$ both represent $2_{\mathbb{Z}}$, and $(0, 1)$ and $(4, 5)$ both represent $-1_{\mathbb{Z}}$.

- (a) Compute $2_{\mathbb{Z}} +_{\mathbb{Z}} (-1_{\mathbb{Z}})$ twice, once with each choice of representatives, and check the answers agree. [2]
- (b) Do the same for $2_{\mathbb{Z}} \times_{\mathbb{Z}} (-1_{\mathbb{Z}})$, showing both products in full. [3]
- (c) Someone defines $f([(p, q)]) = p$. Show with two representatives of one integer that this is not well-defined. [2]

10

7 marks

Each rule below is given using a representative (p, q) of an integer. Decide whether it is well-defined. If it is, say which integer it gives in terms of the input; if not, give two representatives that disagree.

- (a) $g([(p, q)]) = p + q$, a natural number [2]
- (b) $h([(p, q)]) = [(q + 5, p)]$, an integer [3]
- (c) $k([(p, q)]) = [(p, 0)]$, an integer [2]

11

9 marks

Exam style. On $X = \mathcal{P}(\llbracket 3 \rrbracket)$, let AEB when $|A| = |B|$.

- (a) Explain briefly why E is an equivalence relation. [2]
- (b) List the equivalence classes. [2]
- (c) Find $|X/E|$. [1]
- (d) Let $f(A) = |A|^2$. Is f constant on each equivalence class, so that it gives a well-defined f_E ? If so, find the value of f_E on each of the four classes. [3]
- (e) Let $g(A) = 1$ if $0 \in A$ and $g(A) = 0$ otherwise. Does g induce a function on X/E ? [1]

12

9 marks

Exam style. Let $x = [(2, 7)]$ and $y = [(6, 1)]$ in $\mathbb{Z}$.

- (a) Write x and y as $k_{\mathbb{Z}}$ or $-k_{\mathbb{Z}}$. [2]
- (b) Compute $x +_{\mathbb{Z}} y$. [1]
- (c) Compute $x \times_{\mathbb{Z}} y$, and write the answer as $k_{\mathbb{Z}}$ or $-k_{\mathbb{Z}}$. [3]
- (d) Compute $x \times_{\mathbb{Z}} x$ using the representative $(2, 7)$. [2]
- (e) Check your answer to (d) using a different representative of x . [1]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.