

Topic 7 · Sheet A: Worked answers and mark scheme

Partitions, equivalence classes and the integers

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 6

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Is it a partition?

4 marks

(a) **Yes**. (b) **No**: 1 lies in two parts. (c) **No**: $\emptyset$ is not allowed as a part. (d) **No**: 1 is in no part.

B1 each part, naming the failed condition where relevant

4

Why this question Each wrong example breaks exactly one of the three conditions for a partition: parts non-empty, everything covered, and nothing in two parts.

2 Bar notation

4 marks

(a) $03|1|24$ (parts in any order). (b) $\{\{0\}, \{1,2\}, \{3,4\}\}$. (c) **Yes**: the parts are the same sets. (d) **No**: 0 shares a part with 1 in one and with 2 in the other.

B1 each part correct

4

Why this question Bar notation is only shorthand. The partition is still a set of sets, so the order of parts, and of elements within a part, does not matter.

3 Counting partitions of $\llbracket 4 \rrbracket$

6 marks

- (a) **7**: three parts of sizes $3 + 1$, namely $012|3$, $013|2$, $023|1$, $0|123$; and three of sizes $2 + 2$, namely $01|23$, $02|13$, $03|12$.
- (b) **6**: one part has two elements and the others are singletons, and there are 6 ways to choose the pair.
- (c) **1**: the partition 0123 .

M1	(a) splitting into the shapes $3 + 1$ and $2 + 2$	1
A2	(a) all seven listed. <i>A1 for five or six correct, no repeats</i>	2
M1	(b) choosing which pair shares a part	1
A1	(b) 6	1
B1	(c) 1	1

Common mistake Listing $01|23$ and $23|01$ separately. These are the same partition.

4 Equivalence classes from a rule

6 marks

- (a) $[0]_E = \{0, 3, 6\}$ and $[4]_E = \{1, 4, 7\}$.
- (b) $036|147|25$.
- (c) **3**.
- (d) $q_E(7) = \{1, 4, 7\}$. The x with $q_E(x) = [2]_E$ are **2 and 5**.

B1	(a) $[0]_E$	1
B1	(a) $[4]_E$	1
B1	(b)	1
B1	(c) 3	1
B1	(d) $q_E(7)$	1
B1	(d) 2 and 5	1

Why this question Part (d) shows what the quotient map does: it forgets which element you started with and remembers only its class. Different elements in the same class have the same image.

5 From a partition to a relation

6 marks

- (a) **1Q2: yes. 2Q3: no.**
- (b) A part of size s gives s^2 pairs, so $1 + 4 + 9 = 14$.
- (c) $\{3, 4, 5\}$.

- (d) Singletons give the identity relation, with 6 pairs; one part gives the universal relation, with 36.

B1	(a) both	1
M1	(b) s^2 pairs per part	1
A1	(b) 14	1
B1	(c)	1
B1	(d) 6	1
B1	(d) 36	1

Why this question This is the map that turns a partition back into an equivalence relation.

6 Counting via partitions

4 marks

- (a) Equivalence relations correspond to partitions, so this is the count from question 3(a): 7. (b) 3: 01|23, 02|13, 03|12. (c) One class has two elements and the rest are singletons: choose the pair, 10 ways.

M1	(a) using the match with partitions	1
A1	(a) 7	1
B1	(b) 3	1
B1	(c) 10	1

Why this question Counting equivalence relations directly is hard. Counting partitions is much easier, and the two numbers are always equal.

7 Working in $\mathbb{N}^2$

6 marks

- (a) $5 + 4 = 9 = 7 + 2$, so yes.
 (b) $3 + 5 = 8 = 0 + 8$, so yes.
 (c) For example $(2, 0), (3, 1), (4, 2), (5, 3)$.
 (d) $4 - 9 = -5$, so $-5_{\mathbb{Z}}$.
 (e) Elements of $[(0, 4)]_E$ have the form $(k, k + 4)$. Then $2k + 4 = 10$ gives $k = 3$: $(3, 7)$.

B1	(a)	1
B1	(b)	1
B1	(c) four correct pairs	1
B1	(d)	1
M1	(e) the form $(k, k + 4)$, or an equivalent equation	1
A1	(e) $(3, 7)$	1

Why this question Each class lies on a diagonal line of slope one, and (p, q) “means” $p - q$. Part (e) asks for the point on that line where $p + q = 10$.

8 Arithmetic with representatives

6 marks

Note $[(5, 2)] = 3_{\mathbb{Z}}$ and $[(1, 4)] = -3_{\mathbb{Z}}$.

- (a) $[(5 + 1, 2 + 4)] = [(6, 6)] = 0_{\mathbb{Z}}$.
- (b) $[(5 \cdot 1 + 2 \cdot 4, 5 \cdot 4 + 2 \cdot 1)] = [(13, 22)] = -9_{\mathbb{Z}}$. As expected, $3 \times (-3) = -9$.
- (c) $[(2, 5)] = -3_{\mathbb{Z}}$.
- (d) $[(3 \cdot 0 + 0 \cdot 2, 3 \cdot 2 + 0 \cdot 0)] = [(0, 6)] = -6_{\mathbb{Z}}$.

M1	(a) adding entries	1
A1	(a) $0_{\mathbb{Z}}$	1
M1	(b) $(pr + qs, ps + qr)$	1
A1	(b) $-9_{\mathbb{Z}}$	1
B1	(c)	1
B1	(d)	1

Common mistake In (b), multiplying entrywise to get $(5, 8)$. The product formula mixes the entries, as in $(p - q)(r - s) = (pr + qs) - (ps + qr)$.

9 Well-definedness by hand

7 marks

- (a) $(5, 3) + (0, 1) = (5, 4)$ and $(9, 7) + (4, 5) = (13, 12)$. Both represent $1_{\mathbb{Z}}$.
- (b) $(5, 3) \times (0, 1) = (3, 5)$ and $(9, 7) \times (4, 5) = (36 + 35, 45 + 28) = (71, 73)$. Both represent $-2_{\mathbb{Z}}$.
- (c) $(1, 0)$ and $(2, 1)$ both represent $1_{\mathbb{Z}}$, but f gives 1 and 2, so f is not well-defined.

B1	(a) both sums	1
B1	(a) both equal $1_{\mathbb{Z}}$	1
M1	(b) correct product formula	1
A1	(b) both products	1
A1	(b) both equal $-2_{\mathbb{Z}}$	1
M1	(c) two representatives of the same integer	1
A1	(c) showing f disagrees on them	1

Why this question The operations on $\mathbb{Z}$ never depend on the representatives. Part (c) shows that this is a real condition: a rule that “looks inside” a class can easily depend on which element you happen to pick.

10 Which rules are well-defined?

7 marks

- (a) **Not well-defined:** $(1, 0)$ and $(2, 1)$ both represent $1_{\mathbb{Z}}$, but give 1 and 3.
- (b) **Well-defined.** If $p + s = r + q$, then $(q + 5) + r = (s + 5) + p$, so equivalent inputs give equivalent outputs. It is $h(x) = 5_{\mathbb{Z}} - x$, since $(q + 5) - p = 5 - (p - q)$.
- (c) **Not well-defined:** $(1, 0)$ and $(2, 1)$ give $1_{\mathbb{Z}}$ and $2_{\mathbb{Z}}$.

B1	(a) not well-defined	1
B1	(a) two disagreeing representatives	1
B1	(b) well-defined	1
M1	(b) the check that equivalent inputs give equivalent outputs	1
B1	(b) $5_{\mathbb{Z}} - x$	1
B1	(c) not well-defined	1
B1	(c) two disagreeing representatives	1

Why this question The test is always the same: take two equivalent inputs and see whether the outputs are equal (or equivalent). Shifting a representative along its diagonal, from (p, q) to $(p + 1, q + 1)$, is the quickest way to find a counterexample.

11 Exam style: grouping subsets by size

9 marks

- (a) AEB says $|A| = |B|$, and equality of numbers is reflexive, symmetric and transitive, so E is too.
- (b) $\{\emptyset\}, \{\{0\}, \{1\}, \{2\}\}, \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}, \{\{0, 1, 2\}\}$.
- (c) **4**, one class for each size 0, 1, 2, 3.
- (d) **Yes:** $f(A)$ depends only on $|A|$. On the classes of sizes 0, 1, 2, 3, f_E takes the values 0, 1, 4, 9.
- (e) **No:** $\{0\}E\{1\}$, but $g(\{0\}) = 1$ and $g(\{1\}) = 0$.

E2	(a) a reason for each property, or via equality of numbers. <i>E1 if one property is missing</i>	2
B2	(b) all four classes. <i>B1 for three</i>	2
B1	(c) 4	1
B1	(d) yes, with reason	1
B2	(d) the four values. <i>B1 for two or three</i>	2
B1	(e) no, with a pair	1

Why this question This is the well-definedness test in miniature. A function on a quotient is fine exactly when it cannot tell apart elements of the same class. f only looks at the size; g looks at the elements themselves.

12 Exam style: computing in $\mathbb{Z}$

9 marks

- (a) $x = -5\mathbb{Z}$ (since $2 - 7 = -5$) and $y = 5\mathbb{Z}$.
- (b) $[(8, 8)] = 0\mathbb{Z}$.
- (c) $[(2 \cdot 6 + 7 \cdot 1, 2 \cdot 1 + 7 \cdot 6)] = [(19, 44)] = -25\mathbb{Z}$.
- (d) $[(2 \cdot 2 + 7 \cdot 7, 2 \cdot 7 + 7 \cdot 2)] = [(53, 28)] = 25\mathbb{Z}$.
- (e) With $(0, 5)$: $[(0 + 25, 0 + 0)] = [(25, 0)] = 25\mathbb{Z}$. **It agrees.**

B1	(a) x	1
B1	(a) y	1
B1	(b)	1
M1	(c) correct product formula	1
A1	(c) $[(19, 44)]$	1
A1	(c) $-25\mathbb{Z}$	1
M1	(d) correct product formula	1
A1	(d) $25\mathbb{Z}$	1
B1	(e)	1

Why this question Part (d) is $(-5) \times (-5) = 25$, worked out with no rule about signs at all. This is the point: “minus times minus is plus” falls out of the construction.

Total: 74 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.