

Topic 6 · Sheet A: Questions

Equivalence relations, partial orders and total orders

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 5

How to use this sheet. Work on your own paper. The questions start gently and get harder; questions 11 and 12 are exam style. Marks are a guide to how much working is expected, roughly one minute per mark. You need reflexive, symmetric, transitive and antisymmetric relations, equivalence relations, partial orders and total orders, plus Topics 1 to 5.

Notation. A relation R on X is a subset of X^2 , and xRy means $(x, y) \in R$. To show a property fails, one counterexample is enough. An equivalence relation is reflexive, symmetric and transitive; a partial order is reflexive, antisymmetric and transitive; a total order is a partial order in which every two elements are related one way or the other.

1

4 marks

Let $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$, a relation on $\mathbb{Z}$.

- (a) Is R reflexive? [1]
- (b) Is R symmetric? [1]
- (c) Is R transitive? [1]
- (d) Is R an equivalence relation? [1]

2

4 marks

Let $S = \{(0, 1), (1, 2), (0, 2)\}$, a relation on $\mathbb{Z}$. Decide whether S is each of the following, giving a reason.

- (a) reflexive [1]
- (b) symmetric [1]
- (c) transitive [1]
- (d) antisymmetric [1]

3

6 marks

For each relation on $\mathbb{N}$, decide whether it is reflexive, symmetric and transitive. Give a counterexample for each property that fails.

- (a) xRy when $x + y$ is even [3]

- (b) xSy when $x \leq y + 1$ [3]

4

6 marks

Let $R = \{(0,1), (1,2)\}$, a relation on $\llbracket 3 \rrbracket$. In each part, find the smallest relation on $\llbracket 3 \rrbracket$ that contains R and has the given property, by listing the pairs you must add.

- (a) reflexive [1]
 (b) symmetric [1]
 (c) transitive [1]
 (d) an equivalence relation [2]
 (e) How many pairs does your answer to (d) contain? [1]

5

6 marks

Count the relations on $\llbracket 3 \rrbracket$ that are:

- (a) reflexive [1]
 (b) symmetric [2]
 (c) both reflexive and symmetric [1]
 (d) equivalence relations. List them. [2]

6

5 marks

Each relation below is on $\llbracket 3 \rrbracket$, and each contains the identity relation $\Delta = \{(0,0), (1,1), (2,2)\}$. Decide whether each is a partial order.

- (a) $P_1 = \Delta$ [1]
 (b) $P_2 = \Delta \cup \{(0,1), (1,0)\}$ [1]
 (c) $P_3 = \Delta \cup \{(0,1), (1,2)\}$ [1]
 (d) $P_4 = \Delta \cup \{(0,1), (0,2)\}$ [1]
 (e) Which of $P_1, \dots, P_4$ are total orders? [1]

7

6 marks

Consider the boolean poset $(\mathcal{P}(X), \subset)$.

- (a) For $X = \llbracket 2 \rrbracket$, how many pairs (P, Q) satisfy $P \subset Q$? List them. [2]
 (b) Is $\subset$ a total order on $\mathcal{P}(\llbracket 2 \rrbracket)$? [1]
 (c) For $X = \llbracket 3 \rrbracket$, how many pairs (P, Q) satisfy $P \subset Q$? [2]
 (d) Find an element S of $\mathcal{P}(\llbracket 3 \rrbracket)$ with $S \subset T$ for every T , and one with $T \subset S$ for every T . [1]

8

5 marks

Let $D = \{1, 2, 3, 4, 6, 12\}$, and let xRy when y is a multiple of x .

- (a) Is $2R12$? Is $4R6$? [1]
- (b) How many ordered pairs are in R ? [2]
- (c) Is R a total order? Explain. [1]
- (d) List the $x \in D$ with $4Rx$ and $xR12$. [1]

9

7 marks

- (a) How many total orders are there on $\llbracket 3 \rrbracket$? Explain your count. [3]
- (b) Is $x \geq y$ a total order on $\mathbb{N}$? [1]
- (c) Is $\subset$ a total order on $\{\emptyset, \{0\}, \{0, 1\}\}$? [1]
- (d) Is $\subset$ a total order on $\{\emptyset, \{0\}, \{1\}\}$? [1]
- (e) Is $x < y$ a partial order on $\mathbb{N}$? [1]

10

6 marks

Give an example of a relation on $\llbracket 3 \rrbracket$ that is:

- (a) reflexive and symmetric, but not transitive [2]
- (b) symmetric and transitive, but not reflexive [2]
- (c) a partial order, but not a total order [2]

11

10 marks

Exam style. Define a relation on $\llbracket 6 \rrbracket$ by $x \sim y$ when $x + y$ is even.

- (a) Check that $\sim$ is reflexive, symmetric and transitive. [3]
- (b) List every y with $0 \sim y$, and every y with $1 \sim y$. [2]
- (c) How many ordered pairs are in $\sim$? Explain your count. [3]
- (d) Is $\sim$ antisymmetric? [1]
- (e) What is the smallest equivalence relation on $\llbracket 6 \rrbracket$ containing $\sim$ and the pair $(0, 1)$? [1]

12

9 marks

Exam style. On $\mathcal{P}(\llbracket 2 \rrbracket)$, let ARB when $|A| \leq |B|$.

- (a) Decide whether R is reflexive, symmetric and transitive, giving a reason or a counterexample for each. [4]
- (b) Is R antisymmetric? Give a reason. [2]
- (c) Is R total? [1]
- (d) Is R a partial order? Is it a total order? [2]

End of questions. Worked answers and the mark scheme are in the matching answers PDF.