

Topic 6 · Sheet A: Worked answers and mark scheme

Equivalence relations, partial orders and total orders

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 5

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 Checking the three properties

4 marks

(a) **Yes:** all of $(0,0)$, $(1,1)$, $(2,2)$ are in R . (b) **Yes:** the only non-diagonal pairs, $(0,1)$ and $(1,0)$, come together. (c) **Yes:** the only chains are $(0,1)$, $(1,0)$, needing $(0,0)$, and $(1,0)$, $(0,1)$, needing $(1,1)$; both are present. (d) **Yes.**

B1 each part correct

4

Why this question To check transitivity by hand, look only at pairs that chain: xRy and yRz . Pairs with $x = y$ or $y = z$ never cause trouble.

2 A strict-looking relation

4 marks

(a) **No:** $(0,0) \notin S$. (b) **No:** $(0,1) \in S$ but $(1,0) \notin S$. (c) **Yes:** the only chain is $(0,1)$, $(1,2)$, and $(0,2) \in S$. (d) **Yes:** no pair (x,y) with $x \neq y$ appears in both orders.

B1 each part, with a reason

4

Why this question This is the relation $x < y$ on $\llbracket 3 \rrbracket$. It is transitive and antisymmetric but not reflexive, so it is not a partial order.

3 Relations on $\mathbb{N}$

6 marks

- (a) **Reflexive:** $x + x = 2x$ is even. **Symmetric:** $x + y = y + x$. **Transitive:** if $x + y$ and $y + z$ are even, then $x + z = (x + y) + (y + z) - 2y$ is even.
- (b) **Reflexive:** $x \leq x + 1$. **Not symmetric:** 0S5 since $0 \leq 6$, but not 5S0 since $5 > 1$. **Not transitive:** 2S1 and 1S0, but not 2S0, since $2 > 1$.

B1	(a) each property correct	3
B1	(b) reflexive	1
B1	(b) not symmetric, with counterexample	1
B1	(b) not transitive, with counterexample	1

Common mistake Saying a property fails without a specific counterexample. "It is not symmetric" needs a particular pair, such as 0S5 without 5S0.

4 Smallest relations with a property

6 marks

- (a) Add $(0,0), (1,1), (2,2)$.
- (b) Add $(1,0), (2,1)$.
- (c) Add $(0,2)$, since $0R1$ and $1R2$.
- (d) Everything ends up related: **the universal relation** $\mathbb{[3]}^2$. Reflexivity adds the diagonal, symmetry the reverses, and transitivity then links 0 to 2 and 2 to 0.
- (e) 9.

B1	(a)	1
B1	(b)	1
B1	(c)	1
M1	(d) applying all three properties together	1
A1	(d) the universal relation	1
B1	(e) 9	1

Why this question Part (d) shows how fast equivalence relations grow: linking 0 to 1 and 1 to 2 forces all three elements together. Topic 7 explains this using equivalence classes.

5 Counting relations with properties

6 marks

- (a) The 3 diagonal pairs are forced; the other 6 are free: $2^6 = 64$.
- (b) The 3 diagonal pairs are free, and the 3 off-diagonal pairs $\{(x,y), (y,x)\}$ go in or out together: $2^3 \times 2^3 = 64$.

(c) Diagonal forced, 3 off-diagonal pairs free: $2^3 = 8$.

(d) **5**: Δ ; $\Delta \cup \{(0, 1), (1, 0)\}$; $\Delta \cup \{(0, 2), (2, 0)\}$; $\Delta \cup \{(1, 2), (2, 1)\}$; and $[[3]]^2$.

B1	(a) 64	1
M1	(b) treating off-diagonal pairs as linked couples	1
A1	(b) 64	1
B1	(c) 8	1
B1	(d) 5	1
B1	(d) all five listed	1

Why this question Of the 8 reflexive symmetric relations, only 5 are transitive. The three that fail contain two linked couples, such as $0 \sim 1$ and $1 \sim 2$, without the third.

6 Partial orders on $[[3]]$

5 marks

(a) **Yes**. (b) **No**: not antisymmetric, since $0P_21$ and $1P_20$ with $0 \neq 1$. (c) **No**: not transitive, since $(0, 2)$ is missing. (d) **Yes**. (e) **None**: in P_1 and P_4 , the elements 1 and 2 are not related either way.

B1	each part correct	5
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Why this question P_4 says “0 is below everything”, with 1 and 2 side by side. Drawing it as a picture, 0 at the bottom and 1, 2 above, makes it obvious it is partial but not total.

7 The boolean poset

6 marks

(a) **9**: $(\emptyset, \emptyset), (\emptyset, \{0\}), (\emptyset, \{1\}), (\emptyset, \{0, 1\}), (\{0\}, \{0\}), (\{0\}, \{0, 1\}), (\{1\}, \{1\}), (\{1\}, \{0, 1\}), (\{0, 1\}, \{0, 1\})$.

(b) **No**: neither $\{0\} \subset \{1\}$ nor $\{1\} \subset \{0\}$.

(c) $3^3 = 27$: each element of $[[3]]$ is in neither set, in Q only, or in both.

(d) $S = \emptyset$ for the first and $S = [[3]]$ for the second.

B1	(a) 9	1
B1	(a) full list	1
B1	(b) no, with the pair $\{0\}, \{1\}$	1
M1	(c) three options per element	1
A1	(c) 27	1
B1	(d) both	1

Why this question The count in (c) uses the same element-by-element idea as Topic 5, question 10. Here the option “in P but not Q ” is forbidden.

8 Divisibility as a partial order

5 marks

- (a) **2R12: yes. 4R6: no.**
- (b) Count multiples within D : 1 has 6, 2 has 4, 3 has 3, 4 has 2, 6 has 2, 12 has 1. Total **18**.
- (c) **No**: 4 and 6 are not related either way.
- (d) **4 and 12**.

B1	(a) both	1
M1	(b) counting multiples for each x	1
A1	(b) 18	1
B1	(c) no, with a pair	1
B1	(d)	1

Why this question Divisibility is one of the standard partial orders, along with $\leq$ and $\subset$. It is a natural example of a non-total order: most pairs of numbers do not divide each other.

9 Total orders

7 marks

- (a) **6**, one for each way of arranging 0, 1, 2 in a line from smallest to largest.
- (b) **Yes**: it is $\leq$ read backwards, so it is reflexive, antisymmetric, transitive and total.
- (c) **Yes**: $\emptyset \subset \{0\} \subset \{0, 1\}$ is a chain.
- (d) **No**: $\{0\}$ and $\{1\}$ are not related.
- (e) **No**: it is not reflexive.

M1	(a) linking total orders to arrangements in a line	1
M1	(a) counting $3 \times 2 \times 1$	1
A1	(a) 6	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e)	1

Why this question A total order on a finite set is a ranking. So total orders on $\llbracket n \rrbracket$ correspond to permutations, and there are $n!$ of them.

10 Building examples

6 marks

Here $\Delta = \{(0,0), (1,1), (2,2)\}$. Other correct examples score in full.

- (a) $\Delta \cup \{(0,1), (1,0), (1,2), (2,1)\}$: not transitive, since $(0,2)$ is missing.
- (b) $\emptyset$, or $\{(0,0)\}$: symmetric and transitive, but $(1,1)$ is missing.
- (c) Δ : a partial order, but 0 and 1 are not related.

B2 each correct example. *B1 if one property is wrong*

6

Common mistake In (b), adding more pairs “to be safe”. Any pair (x, y) with $x \neq y$ forces (y, x) by symmetry, then (x, x) and (y, y) by transitivity, so small is easier.

11 Exam style: same parity

10 marks

- (a) **Reflexive**: $x + x$ is even. **Symmetric**: $x + y = y + x$. **Transitive**: $x + z = (x + y) + (y + z) - 2y$ is even when $x + y$ and $y + z$ are.
- (b) $0 \sim y$ for $y = 0, 2, 4$; $1 \sim y$ for $y = 1, 3, 5$.
- (c) Each of the two groups of three gives $3^2 = 9$ pairs: **18**.
- (d) **No**: $0 \sim 2$ and $2 \sim 0$, with $0 \neq 2$.
- (e) **The universal relation** $\llbracket 6 \rrbracket^2$: the pair $(0, 1)$ links the evens to the odds.

B1	(a) each property	3
B1	(b) list for 0	1
B1	(b) list for 1	1
M1	(c) splitting into the evens and the odds	1
M1	(c) 3^2 pairs from each group	1
A1	(c) 18	1
B1	(d) no, with a pair	1
B1	(e)	1

Why this question The relation splits $\llbracket 6 \rrbracket$ into evens and odds. Seeing an equivalence relation as a way of grouping elements is exactly the idea of Topic 7.

12 Exam style: comparing sizes of subsets

9 marks

- (a) **Reflexive**: $|A| \leq |A|$. **Not symmetric**: $\emptyset R \{0\}$ but not $\{0\} R \emptyset$. **Transitive**: $|A| \leq |B| \leq |C|$ gives $|A| \leq |C|$.
- (b) **No**: $\{0\} R \{1\}$ and $\{1\} R \{0\}$, but $\{0\} \neq \{1\}$.

(c) **Yes**: for any two sets one size is at most the other.

(d) **Not a partial order**, since it is not antisymmetric, and so **not a total order** either.

B1	(a) reflexive	1
B1	(a) not symmetric	1
E1	(a) a counterexample pair such as $\emptyset$, $\{0\}$	1
B1	(a) transitive, with reason	1
B1	(b) no	1
E1	(b) a valid pair	1
B1	(c)	1
B1	(d) not a partial order	1
B1	(d) not a total order	1

Common mistake Concluding “total, so a total order”. A total order must first be a partial order, and this relation fails antisymmetry.

Total: 74 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.