

Topic 5 · Sheet A: Worked answers and mark scheme

Union, intersection, difference and the boolean algebra of sets

Suggested time

About 75 minutes

Total marks

74

Questions

12

Builds on

Topics 1 to 4

How marks are awarded. Each mark has a code, as on most UK exam papers.

B A mark for a correct statement or value on its own. It does not depend on any other mark.

M A method mark, for a correct approach, even if the final answer is wrong.

A An accuracy mark, for the correct answer *after* a correct method. It cannot be awarded without the M mark before it.

E An explanation mark, for a clear reason in words. A correct reason that is worded differently scores in full.

For true or false questions, the mark is for the correct decision; where a reason is asked for, it must be the right reason.

1 The four basic operations

4 marks

(a) $\{0, 1, 2, 3, 4\}$ (b) $\{2, 3\}$ (c) $\{0, 1\}$ (d) $\{4\}$

B1 each part correct

4

Why this question Parts (c) and (d) differ, so set difference is not commutative, just like subtraction of numbers.

2 Operations on $\llbracket n \rrbracket$

4 marks

(a) $\llbracket 5 \rrbracket$ (b) $\llbracket 3 \rrbracket$ (c) $\{3, 4\}$ (d) $\emptyset$

B1 each part correct

4

Why this question The nested sets $\llbracket n \rrbracket$ give $\llbracket m \rrbracket \cup \llbracket n \rrbracket = \llbracket \max(m, n) \rrbracket$; the same picture gives $\llbracket m \rrbracket \cap \llbracket n \rrbracket = \llbracket \min(m, n) \rrbracket$. These nested sets make good mental models.

3 Laws for the empty set

5 marks

- (a) X (b) $\emptyset$ (c) X (d) $\emptyset$ (e) $\emptyset$

B1 each part correct

5

Why this question These are the identity laws for union, intersection and difference. $\emptyset$ behaves like 0: adding it changes nothing, and multiplying by it gives nothing.

4 Sets given by properties

6 marks

Here $A = \{0, 2, 4, \dots, 18\}$, $B = \{0, 3, 6, 9, 12, 15, 18\}$ and $C = \{0, 1, 2, 3, 4, 5\}$.

- (a) $\{0, 6, 12, 18\}$
(b) $\{3, 9, 15\}$
(c) $\{0, 1, 2, 3, 4, 5, 6, 12, 18\}$
(d) $A \cup B$ contains 0, 2, 3, 4 from C , so $C - (A \cup B) = \{1, 5\}$.
(e) $|A| = 10$, $|B| = 7$, $|A \cup B| = 13$, and $10 + 7 - 4 = 13$. **It checks.**

B1	(a)	1
B1	(b)	1
B1	(c)	1
B1	(d)	1
B1	(e) all three sizes	1
B1	(e) the check	1

Why this question Part (e) shows why $|A \cup B|$ is not $|A| + |B|$: the elements of $A \cap B$ are counted twice, so they are subtracted once.

5 Unions and intersections over a set of sets

6 marks

- (a) Union $\llbracket 4 \rrbracket$; intersection $\llbracket 1 \rrbracket = \{0\}$.
(b) Union $\{0, 1, 2, 3\}$; intersection $\emptyset$, since no element is in all three.
(c) Union $\mathbb{N}$, since every n lies in $\llbracket n + 1 \rrbracket$; intersection $\emptyset$, since $\llbracket 0 \rrbracket = \emptyset$ is one of the sets.

B1	each union	3
B1	each intersection	3

Common mistake In (b), saying the intersection is $\{1, 2\}$ because neighbouring sets overlap. An element of the intersection must lie in *every* set in $\mathcal{X}$.

6 Disjoint sets and absorption

5 marks

(a) **Yes**. (b) **No**: 0 and 6 are in both. (c) $7 + 5 - 2 = 10$. (d) $Y \subset X$, by absorption. (e) $X \subset Y$, by absorption for intersection.

B1	each part correct	5
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Why this question The absorption laws in (d) and (e) let you trade a statement about $\cup$ or $\cap$ for a statement about $\subset$. Many later arguments use this switch.

7 De Morgan's laws on an example

6 marks

- (a) $\{2\}$
(b) $\{0, 4, 5, 6\}$. Here $X \cup Y = \{0, 1, 2, 3, 4, 5, 6, 8\}$.
(c) Both are $\{5\}$. This is **De Morgan's first law**, $X - (Y \cup Z) = (X - Y) \cap (X - Z)$.
(d) Both are $\{0, 1, 3, 4, 5\}$, the second De Morgan law.

B1	(a)	1
B1	(b)	1
B1	(c) both sets	1
B1	(c) naming De Morgan's law	1
B1	(d) $X - (Y \cap Z)$	1
B1	(d) $(X - Y) \cup (X - Z)$	1

Why this question De Morgan's laws swap $\cup$ and $\cap$ when you take a difference. Checking them on a small example is the fastest way to remember which way round they go.

8 The distributive laws, and a non-law

6 marks

- (a) Both equal $\{1, 2\}$.
(b) Both equal $\{0, 1, 2, 3\}$.
(c) $(X - Y) - Z = \{0\}$ but $X - (Y - Z) = \{0, 2\}$. So **set difference is not associative**.

B1	(a) first set	1
B1	(a) second set	1
B1	(b) first set	1
B1	(b) second set	1
B1	(c) both sets	1
E1	(c) not associative	1

Why this question Union and intersection are associative, but difference is not. A single example is enough to show a law fails.

9 Operations on power sets

7 marks

- (a) $\{\emptyset, \{1\}\} = \mathcal{P}(\{1\})$, and $\{1\} = A \cap B$.
 (b) $\{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{2\}, \{1, 2\}\}$, six elements.
 (c) **No.** $A \cup B = \{0, 1, 2\}$, so $\{0, 2\} \in \mathcal{P}(A \cup B)$, but $\{0, 2\}$ is a subset of neither A nor B .

B1	(a) the two elements	1
B1	(a) $\mathcal{P}(\{1\})$, or $\mathcal{P}(A \cap B)$	1
B2	(b) all six, once each. <i>B1 for five correct, no wrong entries</i>	2
B1	(c) no	1
B1	(c) $A \cup B = \{0, 1, 2\}$	1
E1	(c) a subset such as $\{0, 2\}$ that is missing	1

Why this question Intersections of power sets behave well: $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ always. Unions do not, because a subset of $A \cup B$ can mix elements from both.

10 Counting with unions and intersections

6 marks

- (a) S must avoid 0 and 1, so it is any subset of $\{2, 3, 4, 5\}$: **16**.
 (b) S must contain 2, 3, 4, 5, and 0, 1 are free: **4**.
 (c) Each element of $\llbracket 3 \rrbracket$ goes in S , in T , or in neither (not both): 3 choices each, so $3^3 = 27$.

B1	(a) 16	1
M1	(b) seeing 2, 3, 4, 5 are forced	1
A1	(b) 4	1
M1	(c) three options per element	1
M1	(c) multiplying over the three elements	1
A1	(c) 27	1

Why this question Part (c) is a clever count: instead of choosing S then T , decide where each element goes. This element-by-element view is used again and again.

11 Exam style: sets inside $\llbracket 12 \rrbracket$

10 marks

Here $A = \{0, 2, 4, 6, 8, 10\}$ and $B = \{7, 8, 9, 10, 11\}$.

- (a) $A \cap B = \{8, 10\}$ and $A \cup B = \{0, 2, 4, 6, 7, 8, 9, 10, 11\}$.

- (b) Both are $\{1, 3, 5\}$. They agree by De Morgan's law: $U - (A \cup B) = (U - A) \cap (U - B)$.
- (c) $\{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, with cardinality **11**. Only 1 is missing.
- (d) $A - B = \{0, 2, 4, 6\}$ and $B - A = \{7, 9, 11\}$, so $\{0, 2, 4, 6, 7, 9, 11\}$. Removing 8 and 10 from $A \cup B$ gives **the same set**.

B1	(a) $A \cap B$	1
B1	(a) $A \cup B$	1
B1	(b) $U - (A \cup B)$	1
B1	(b) $(U - A) \cap (U - B)$	1
E1	(b) De Morgan's law	1
B1	(c) the set	1
B1	(c) 11	1
M1	(d) finding $A - B$ and $B - A$	1
A1	(d)	1
B1	(d) the check	1

Why this question The set in (d) is the elements in exactly one of A and B . It equals $(A \cup B) - (A \cap B)$, which you can check from part (a).

12 Exam style: simplifying set expressions

9 marks

With $X = \{0, 1, 2\}$ and $Y = \{1, 2, 3\}$: $X \cap Y = \{1, 2\}$, $X \cup Y = \{0, 1, 2, 3\}$, $X - Y = \{0\}$, $Y - X = \{3\}$.

- (a) **X**, since $X \subset X \cup Y$ (absorption). Check: $\{0, 1, 2\} \cap \{0, 1, 2, 3\} = \{0, 1, 2\}$.
- (b) **X**, since $X \cap Y \subset X$. Check: $\{0, 1, 2\} \cup \{1, 2\} = \{0, 1, 2\}$.
- (c) **X**: every element of X is either not in Y or in Y . Check: $\{0\} \cup \{1, 2\} = \{0, 1, 2\}$.
- (d) **$X \cap Y$** : removing from X the elements not in Y leaves those in Y . Check: $\{0, 1, 2\} - \{0\} = \{1, 2\}$.
- (e) **$\emptyset$** : nothing is in X but not Y , and also in Y but not X . Check: $\{0\} \cap \{3\} = \emptyset$.

B1	(a) X	1
B1	(a) check	1
B1	(b) X , with check	1
B1	(c) X	1
B1	(c) check	1
B1	(d) $X \cap Y$	1
E1	(d) general reason	1
B1	(d) check	1
B1	(e) $\emptyset$, with check	1

Common mistake In (d), cancelling the two minus signs to get $X - X + Y$ or similar. Set difference is not subtraction of numbers; think about which elements survive.

Total: 74 marks. Spotted a mistake? Report it at unimixedtutor.com and we will fix it.